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On subordination of holomorphic semigroups ☆

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We prove that for any Bernstein function ψ the operator $-\psi(A)$ generates a bounded holomorphic C_0-semigroup $(e^{-t\psi(A)})_{t\geq 0}$ on a Banach space, whenever $-A$ does. This answers a question posed by Kishimoto and Robinson. Moreover, giving a positive answer to a question by Berg, Boyadzhiev and de Laubenfels, we show that $(e^{-t\psi(A)})_{t\geq 0}$ is holomorphic in the holomorphy sector of $(e^{-tA})_{t\geq 0}$, and if $(e^{-tA})_{t\geq 0}$ is sectorially bounded in this sector then $(e^{-t\psi(A)})_{t\geq 0}$ has the same property. We also obtain new sufficient conditions on ψ in order that, for every Banach space X, the semigroup $(e^{-t\psi(A)})_{t\geq 0}$ on X is holomorphic whenever $(e^{-tA})_{t\geq 0}$ is a bounded C_0-semigroup on X. These conditions improve and generalize well-known results by Carasso–Kato and Fujita.

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1. Introduction

The present paper concerns operator-theoretic and function-theoretic properties of Bernstein functions and solves several notable problems which have been left open for some time.

Bernstein functions play a prominent role in probability theory and operator theory. One of their characterizations, also important for our purposes, says that a function $\psi : (0, \infty) \to [0, \infty)$ is Bernstein if and only if there exists a vaguely continuous semigroup of subprobability Borel measures $(\mu_t)_{t\geq 0}$ on $[0, \infty)$ such that

$$e^{-t\psi(\lambda)} = \int_0^\infty e^{-\lambda s}\, \mu_t(ds), \qquad \lambda > 0, \tag{1.1}$$

for all $t \geq 0$.

Let now $(e^{-tA})_{t\geq 0}$ be a bounded C_0-semigroup on a (complex) Banach space X with generator $-A$. The relation (1.1) suggests a way to define a new bounded C_0-semigroup $(e^{-tB})_{t\geq 0}$ on X in terms of $(e^{-tA})_{t\geq 0}$ and a Bernstein function ψ as

$$e^{-tB} = \int_0^\infty e^{-sA}\, \mu_t(ds), \tag{1.2}$$

where $(\mu_t)_{t\geq 0}$ is a semigroup of measures given by (1.1). Following (1.1), it is natural to define $\psi(A) := B$. As it will be revealed in Subsection 3.3 below, such a definition of $\psi(A)$ goes far beyond formal notation and it respects some rules for operator functions called functional calculus.

The semigroup $(e^{-t\psi(A)})_{t\geq 0}$ is subordinated to the semigroup $(e^{-tA})_{t\geq 0}$ via a subordinator $(\mu_t)_{t\geq 0}$. The basics of subordination theory was set up by Bochner [4] and Phillips [27]. This approach to constructing semigroups is motivated by probabilistic applications, e.g. by the study of Lévy processes, but it has also significant value for PDEs as well. As a textbook example one may mention a classical result of Yosida expressing $(e^{-tA^\alpha})_{t\geq 0}$, $\alpha \in (0, 1)$, in terms of $(e^{-tA})_{t\geq 0}$ as in (1.2), see e.g. [33]. The essential feature of this example is that C_0-semigroups $(e^{-tA^\alpha})_{t\geq 0}$ turn out to be necessarily holomorphic. This fact stimulated further research on relations between functional calculi and Bernstein functions, see e.g. [31] and [32]. Some of them are described below.

An easy consequence of (1.2) is that for a fixed Bernstein function ψ the mapping

$$\mathcal{M} : -A \mapsto -\psi(A) \tag{1.3}$$

preserves the class of generators of bounded C_0-semigroups, and it is natural to ask whether there are any other important classes of semigroup generators stable under $\mathcal{M}$. In particular, whether $\mathcal{M}$ preserves the class of holomorphic C_0-semigroups. The question was originally asked by Kishimoto and Robinson in [18, p. 63, Remark]. It appeared to be quite difficult and there have been very few general results in this direction so far.

A partial answer to the Kishimoto–Robinson question was obtained in [3] where the question was formulated in another form: whether $\mathcal{M}$ preserves the class of sectorially bounded holomorphic C_0-semigroups? It was proved in [3, Theorem 7.2] that for any Bernstein function ψ the operator $-\psi(A)$ generates a sectorially bounded holomorphic C_0-semigroup of angle $\pi/2$, whenever $-A$ does. Moreover, if $-A$ generates a sectorially bounded holomorphic C_0-semigroup of angle greater than $\pi/4$ then $-\psi(A)$ is the generator of a sectorially bounded holomorphic C_0-semigroup as well [3, Proposition 7.4]. However, in the latter case, the relations between the sectors of holomorphy of the two semigroups was not made precise in [3].

An affirmative answer to the Kishimoto–Robinson question for uniformly convex Banach spaces X was obtained in [24] using Kato–Pazy's characterization of holomorphic C_0-semigroups on uniformly convex spaces. In fact, a positive answer to the question in its full generality was also claimed in [21]. However, there seems to be an error in the arguments there (see Remark 4.8 for more on that), and moreover the permanence of sectors and thus the sectorial boundedness of semigroups has not been addressed in [21] and [24].

Another class of problems related to $\mathcal{M}$ concerns Bernstein functions ψ yielding semigroups $(e^{-t\psi(A)})_{t\geq 0}$ with better properties than the initial semigroup $(e^{-tA})_{t\geq 0}$, as in Yosida's example with $\psi(\lambda) = \lambda^\alpha$, $\alpha \in (0, 1)$. In particular, it is of value to know when Bernstein functions transform generators of bounded C_0-semigroups into generators of bounded holomorphic C_0-semigroups. (Here the boundedness of the semigroup is assumed only on the real half-line.) Bernstein functions having this property will further be called *Carasso–Kato functions* since the first general results on their structure are due to Carasso and Kato [6]. In particular, [6, Theorem 4] gives a criterion for a Bernstein function ψ to be Carasso–Kato in terms of the semigroup $(\mu_t)_{t\geq 0}$ corresponding to ψ and also a necessary condition for that property in terms of ψ itself. Note that while a characterization of a Carasso–Kato function ψ in terms of $(\mu_t)_{t\geq 0}$ exists, it can hardly be applied directly since it is, in general, highly nontrivial to construct $(\mu_t)_{t\geq 0}$ corresponding to ψ. Thus it is desirable to have direct characterizations of Carasso–Kato functions.

Certain sufficient conditions for a Bernstein function to be Carasso–Kato were obtained in [6,9,22,23,30]. Interesting applications of Carasso–Kato functions can be found in [5,10,19]. We note also [7] where similar results were obtained in a discrete setting.

Our approach to the two problems on $\mathcal{M}$ mentioned above relies on certain extensions of the theory of Bernstein functions and its applications to operator norm estimates by means of functional calculi. Observe that the problems are comparatively simple if ψ is a *complete* Bernstein function [3]. Thus it is natural to try to use this partial answer in a more general setting of Bernstein functions. Our main idea relies on comparing a fixed Bernstein function ψ to a complete Bernstein function φ *associated* to ψ in a unique way.

It appears that the functions ψ and φ are intimately related and the behavior of ψ and its transforms match in a natural sense the behavior of φ and the corresponding transforms. So our aim is to show that for appropriate z the "resolvent" functions $(z + \psi(\cdot))^{-1}$ and $(z + \varphi(\cdot))^{-1}$ differ by a summand with good integrability (and other analytic) properties and then to recast this fact in terms of functional calculi. The latter step is not however direct and to perform it correctly and transparently we have to use an interplay between several well-known calculi. Apart from answering the questions from [18] and [3], another advantage of our approach is that we have a good control over fine properties of $\psi(A)$, thus deriving the property of permanence of angles under the map $\mathcal{M}$.

Our functional calculus approach leads, in particular, to the following statement which is one of the main results in this paper.

Theorem 1.1. *Let $-A$ be the generator of a bounded holomorphic C_0-semigroup of angle $\theta \in (0, \pi/2]$ on a Banach space X. Then for every Bernstein function ψ the operator $-\psi(A)$ generates a bounded holomorphic C_0-semigroup of angle θ on X as well. Moreover, if $-A$ generates a sectorially bounded holomorphic C_0-semigroup of angle θ, then the same is true for $-\psi(A)$.*

The functional calculus ideas allow one also to *characterize* the Carasso–Kato property of ψ if ψ is a complete Bernstein function, i.e. if, in addition, ψ extends to the upper half-plane and maps it into itself. The characterization given in Corollary 5.8 below is a consequence of the following interesting result (see Theorem 5.7).

Theorem 1.2. *Let ψ be a complete Bernstein function and let $\gamma \in (0, \pi/2)$ be fixed. The following assertions are equivalent.*

- (i) *The function ψ maps the right half-plane into the sector $\Sigma_\gamma := \{\lambda \in \mathbb{C} : |\arg \lambda| < \gamma\}$.*
- (ii) *For each (complex) Banach space X and each generator $-A$ of a bounded C_0-semigroup on X, the operator $-\psi(A)$ generates a sectorially bounded holomorphic C_0-semigroup on X of angle $\pi/2 - \gamma$.*

Moreover, we are able to strengthen essentially the results by Fujita from [9] removing in particular several assumptions made in [9].

Theorem 1.3. *Let ψ be a Bernstein function. Suppose there exist $\theta \in (\pi/2, \pi)$ and $r > 0$ such that ψ admits a holomorphic extension to Σ_θ, and*

$$0 < \arg(\psi(\lambda)) < \pi/2 \qquad \text{if} \qquad 0 < \arg \lambda < \theta \quad \text{and} \quad |\lambda| \geq r. \tag{1.4}$$

If $-A$ is the generator of a bounded C_0-semigroup on a Banach space X, then the (bounded) C_0-semigroup $(e^{-t\psi(A)})_{t\geq 0}$ is holomorphic in Σ_{θ_0} with $\theta_0 = \frac{\pi}{2}(1 - \pi/(2\theta))$.

Let us describe the structure of our paper. The paper is organized as follows. Section 2 contains basic information on Bernstein functions together with new notions and properties which are related to Bernstein functions and are crucial for the sequel. In Section 3, we review functional calculi theory needed for the proofs of our main results and prove several auxiliary statements. Section 4 is devoted to the proof of one of our central results, Theorem 1.1. In Section 5, we study Carasso–Kato functions and complement and strengthen the corresponding statements by Carasso–Kato and Fujita. Finally, in Appendix A, we comment on alternative ways to prove Theorem 1.1.

We finish the introduction with fixing some notation for the rest of the paper. For a closed linear operator A on a complex Banach space X we denote by $\operatorname{dom}(A)$ and $\rho(A)$ the *domain* and the *resolvent set* of A, respectively. If a linear operator A is closable, then we denote its closure by $\overline{A}$. The space of bounded linear operators on X is denoted by $\mathcal{L}(X)$.

The Laplace transform $\widehat{\mu}$ of a Laplace transformable measure μ will be defined as usual as

$$\widehat{\mu}(\lambda) := \int_0^\infty e^{-\lambda s}\,\mu(ds)$$

for appropriate λ. For a set $S \subset \mathbb{C}$, its closure will be denoted by $\overline{S}$.

Finally, we let

$$\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}, \quad H^+ := \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda > 0\}, \quad \text{and} \quad \mathbb{R}_+ := [0,\infty),$$

and for $\beta \in (0,\pi]$, we denote

$$\Sigma_\beta := \{\lambda \in \mathbb{C} : |\arg\lambda| < \beta\}, \quad \Sigma_\beta^+ = \{\lambda \in \mathbb{C} : 0 < \arg\lambda < \beta\}, \quad \text{and} \quad \Sigma_0 := (0,\infty).$$

2. Bernstein functions

This section will lay a function-theoretic background for our functional calculi considerations in the subsequent sections. We will present several properties of Bernstein functions useful for the sequel, and we will show how any Bernstein function can be approximated by a complete Bernstein function. Some of the properties of Bernstein functions proved below are certainly known, but we were not able to find the explicit references to them in the literature. As a general reference for the theory of Bernstein functions we mention [32].

We start by recalling one of possible definitions of a Bernstein function.

Definition 2.1. An infinitely differentiable function $\psi : (0,\infty) \mapsto [0,\infty)$ is called *Bernstein* if its derivative ψ' is *completely monotone*, i.e.

$$\psi'(\lambda) = \int_0^\infty e^{-\lambda s}\, \nu(ds), \qquad \lambda > 0, \tag{2.1}$$

for a Laplace transformable positive Borel measure ν on $[0, \infty)$.

The class of Bernstein functions will be denoted by $\mathcal{BF}$. The standard examples of Bernstein functions include $1 - e^{-\lambda}$, $\log(1 + \lambda)$, and λ^α, $\alpha \in [0, 1]$.

By [32, Theorem 3.2], ψ is Bernstein if and only if there exist $a, b \geq 0$ and a positive Borel measure μ on $(0, \infty)$ satisfying

$$\int_{0+}^\infty \frac{s}{1+s}\, \mu(ds) < \infty$$

such that

$$\psi(\lambda) = a + b\lambda + \int_{0+}^\infty (1 - e^{-\lambda s})\mu(ds), \qquad \lambda > 0, \tag{2.2}$$

where we write the Lebesgue integral $\int_{(0,\infty)}$ as $\int_{0+}^\infty$. The formula (2.2) is called *the Lévy–Hintchine representation* of ψ. The triple (a, b, μ) is defined uniquely and is called Lévy triple of ψ. We will then often write $\psi \sim (a, b, \mu)$ meaning the Lévy–Hintchine representation of ψ. Every Bernstein function extends analytically to $\mathbb{C}_+$ and continuously to $\overline{\mathbb{C}}_+$. In the following Bernstein functions will be identified with their continuous extensions to $\overline{\mathbb{C}}_+$. It is instructive to note that the set $\mathcal{BF}$ is closed under composition [32, Corollary 3.8, (iii)]. There is a profound theory of Bernstein functions with many implications in functional analysis and probability theory. For a comprehensive account of that theory, we refer the reader to a recent book [32].

Geometric properties of Bernstein functions will be of particular importance for us, in particular, the fact that a Bernstein function preserves angular sectors symmetric with respect to $\mathbb{R}_+$, see e.g. [32, Proposition 3.6]. For later use, we state this result as a proposition below.

Proposition 2.2. *Let $\psi \in \mathcal{BF}$. Then for every $\omega \in (0, \pi/2]$,*

$$\psi(\Sigma_\omega) \subset \Sigma_\omega. \tag{2.3}$$

(In fact, (2.3) is a property of all holomorphic functions preserving $\mathbb{C}_+$ and $(0, \infty)$, see e.g. [29, Theorem 2].)

We will also need several simple estimates of Bernstein functions given in the following proposition.

Proposition 2.3. *Let $\psi \in \mathcal{BF}$.*

(i) *For all $\gamma \ge 0$, $\beta \in [0, \pi/2]$ such that $\gamma + \beta < \pi$,*

$$|z + \psi(\lambda)| \ge \cos((\gamma + \beta)/2)\,(|z| + |\psi(\lambda)|), \quad z \in \Sigma_\gamma, \qquad \lambda \in \Sigma_\beta.$$

(ii) *One has*

$$\mathrm{Re}(\psi(\lambda)) \ge \psi(\mathrm{Re}\,\lambda), \qquad \lambda \in \mathbb{C}_+.$$

(iii) *There exists $c_\psi > 0$ such that*

$$|\psi(\lambda)| \le c_\psi |\lambda|, \qquad \lambda \in \overline{\mathbb{C}}_+, \quad |\lambda| \ge 1.$$

(iv) *For all $\beta \in [0, \pi/2)$,*

$$|\psi(\lambda)| \ge |\lambda| \psi'(1) \cos\beta, \qquad \lambda \in \Sigma_\beta, \quad |\lambda| \le 1.$$

Proof. To prove (i) it suffices to observe that

$$|z + \lambda| \ge \cos((\beta + \gamma)/2)\,(|z| + |\lambda|), \quad z \in \Sigma_\gamma, \quad \lambda \in \Sigma_\beta. \tag{2.4}$$

(The inequality above is evident if one considers the vectors z and $-\lambda$, notes that the angle between them is at most $\pi - (\beta+\gamma)$, and drops perpendiculars from their endpoints onto the bisector of the angle.) Then (i) is a direct consequence of Proposition 2.2 and (2.4).

To obtain (ii), note that

$$\mathrm{Re}(1 - e^{-\lambda}) \ge 1 - e^{-\mathrm{Re}\,\lambda}, \quad \lambda \in \mathbb{C}_+,$$

and use the Lévy–Hintchine representation for ψ.

To prove (iii), observe that if $\psi \sim (a, b, \mu)$ then (2.2) yields

$$|\psi(\lambda)| \le a + b|\lambda| + |\lambda| \int_{0+}^{1} s\,\mu(\mathrm{d}s) + 2\int_{1}^{\infty} \mu(\mathrm{d}s), \qquad \lambda \in \overline{\mathbb{C}}_+,$$

and (iii) follows.

Furthermore, since $\psi(s\lambda) \le s\psi(\lambda)$ for $\lambda > 0$ and $s \ge 1$ (see [16, p. 205]), (ii) implies that for any $\beta \in [0, \pi/2)$:

$$|\psi(\lambda)| \ge \psi(|\lambda|) \cos\beta, \qquad \lambda \in \Sigma_\beta. \tag{2.5}$$

Using

$$\lambda\psi'(\lambda) \le \psi(\lambda), \qquad \lambda > 0 \tag{2.6}$$

(see [16, p. 204]), we have

$$\psi(|\lambda|) \ge |\lambda|\psi'(|\lambda|) \ge \psi'(1)|\lambda|, \qquad |\lambda| \in (0, 1]. \tag{2.7}$$

Now (2.5) and (2.7) yield (iv). □

It is often convenient to restrict one's attention to a subclass of Bernstein functions formed by complete Bernstein functions. It has a rich structure which makes it especially useful in applications. A Bernstein function ψ is said to be *a complete Bernstein function* if the measure μ in its Lévy–Hintchine representation (2.2) has a completely monotone density with respect to Lebesgue measure. The set of all complete Bernstein functions will be denoted by $\mathcal{CBF}$.

The class of complete Bernstein functions allows a number of characterizations. The ones relevant for our purposes are summarized in the following statement, see e.g. [32, Theorem 6.2].

Theorem 2.4. *Let φ be a non-negative function on $(0, \infty)$. Then the following conditions are equivalent.*

(i) $\varphi \in \mathcal{CBF}$.

(ii) *There exists a (unique) Bernstein function ψ such that*

$$\varphi(\lambda) = \lambda^2 \widehat{\psi}(\lambda), \quad \lambda > 0. \tag{2.8}$$

(iii) *φ admits a holomorphic extension to H^+ such that*

$$\mathrm{Im}(\varphi(\lambda)) \ge 0 \quad \text{for all} \quad \lambda \in H^+,$$

and such that the limit

$$\varphi(0+) = \lim_{\lambda \to 0+} \varphi(\lambda)$$

exists.

(iv) *φ admits a holomorphic extension to $\mathbb{C} \setminus (-\infty, 0]$ which is given by*

$$\varphi(\lambda) = a + b\lambda + \int_{0+}^{\infty} \frac{\lambda\,\sigma(ds)}{\lambda + s}, \tag{2.9}$$

where $a, b \ge 0$ and σ is a positive Borel measure on $(0, \infty)$ such that

$$\int_{0+}^{\infty} \frac{\sigma(ds)}{1+s} < \infty. \tag{2.10}$$

The triple (a, b, σ) *is defined uniquely and it is called the* Stieltjes representation *of* φ.

Using Theorem 2.4, (iii) it is easy to see that, for instance, λ^α, $\alpha \in [0, 1]$, and $\log(1+\lambda)$ belong to $\mathcal{CBF}$, while $1 - e^{-\lambda} \notin \mathcal{CBF}$. Another implication of this statement is that the composition of complete Bernstein functions is complete Bernstein. Note also that if $\varphi \in \mathcal{CBF}$ then for each $\theta \in [0, \pi)$ there exists $C_\theta > 0$ such that $|\varphi(\lambda)| \le C_\theta|\lambda|$ for all $\lambda \in \Sigma_\theta$ with $|\lambda| \ge 1$. This follows directly from (2.9).

The next statement sharpens Proposition 2.2 in a specific situation when complete Bernstein function has its range in a sector smaller than the right half-plane.

Proposition 2.5. *Let* $\varphi \in \mathcal{CBF}$, $\varphi \not\equiv const$, *and suppose that*

$$\varphi(\mathbb{C}_+) \subset \Sigma_\gamma \tag{2.11}$$

for some $\gamma \in (0, \pi/2)$. *Let* $\theta_0 \in (\pi/2, \pi)$ *be defined by*

$$|\cos\theta_0| = \frac{\cot\gamma}{1+\cot\gamma}. \tag{2.12}$$

Then for every $\theta \in [\pi/2, \theta_0]$ *one has*

$$\varphi(\Sigma_\theta) \subset \Sigma_{\tilde{\theta}},$$

where $\tilde{\theta} \in (0, \pi/2]$ *is given by*

$$\cot\tilde{\theta} := \frac{1+\cot\gamma}{\sin\theta}\left(\frac{\cot\gamma}{1+\cot\gamma} - |\cos\theta|\right). \tag{2.13}$$

Proof. From (2.11) it follows that φ has the Stieltjes representation $(a, 0, \sigma)$. Note that

$$\varphi(re^{i\theta}) = a + \int_{0+}^{\infty} \frac{r(r+t\cos\theta)\,\sigma(dt)}{r^2+t^2+2rt\cos\theta}$$
$$+ i\sin\theta \int_{0+}^{\infty} \frac{rt\,\sigma(dt)}{r^2+t^2+2rt\cos\theta}, \qquad r > 0, \quad |\theta| < \pi, \tag{2.14}$$

and $\mathrm{Im}(\varphi(re^{i\theta})) > 0$ for $r > 0$ and $\theta \in (0, \pi)$. Setting $\theta = \pi/2$ in (2.14) and using (2.11), we infer that

$$a + \int_{0+}^{\infty} \frac{r^2\,\sigma(dt)}{r^2 + t^2} \ge \cot\gamma \int_{0+}^{\infty} \frac{rt\,\sigma(dt)}{r^2 + t^2}, \qquad r > 0. \tag{2.15}$$

Moreover, observe that for every $\theta \in [\pi/2, \pi)$, and all $r, t > 0$,

$$\frac{1}{r^2 + t^2} \le \frac{1}{r^2 + t^2 + 2rt\cos\theta} \le \frac{1}{(1 - |\cos\theta|)(r^2 + t^2)}. \tag{2.16}$$

Hence, if $\theta \in [\pi/2, \theta_0]$, where θ_0 is defined by (2.12), then by (2.14), (2.16) and (2.15) we obtain

$$\begin{aligned}
\mathrm{Re}(\varphi(re^{i\theta})) &\ge a + \int_{0+}^{\infty} \frac{r^2 - rt|\cos\theta|}{r^2 + t^2 + 2rt\cos\theta}\,\sigma(dt) \\
&\ge a + \int_{0+}^{\infty} \frac{r^2\,\sigma(dt)}{r^2 + t^2} - \frac{|\cos\theta|}{1 - |\cos\theta|} \int_{0+}^{\infty} \frac{rt\,\sigma(dt)}{r^2 + t^2} \\
&\ge \left(\cot\gamma - \frac{|\cos\theta|}{1 - |\cos\theta|}\right) \int_{0+}^{\infty} \frac{rt}{r^2 + t^2}\,\sigma(dt) \\
&\ge \left(\cot\gamma - \frac{|\cos\theta|}{1 - |\cos\theta|}\right)(1 - |\cos\theta|) \int_{0+}^{\infty} \frac{rt\,\sigma(dt)}{r^2 + t^2 + 2rt\cos\theta} \\
&= \alpha(\theta)\,\mathrm{Im}(\varphi(re^{i\theta})),
\end{aligned}$$

where $\alpha(\theta)$ is given by the right hand side of (2.13). Note that

$$\alpha(\theta_0) = 0, \quad \alpha(\pi/2) = \cot\gamma \quad \text{and} \quad \alpha(\theta) > 0 \quad \text{if} \quad \theta \in [\pi/2, \theta_0).$$

Moreover,

$$\alpha'(\theta) = \frac{\cot\gamma|\cos\theta| - (1 + \cot\gamma)}{\sin^2\theta} \le -\frac{1}{\sin^2\theta} < 0, \quad \theta \in [\pi/2, \theta_0],$$

hence the function α is positive and decreasing on $[\pi/2, \theta_0]$. Therefore, for all $\theta \in (\pi/2, \theta_0)$ and $\theta' \in (\pi/2, \theta)$ we have

$$\mathrm{Re}(\varphi(re^{i\theta'})) \ge \alpha(\theta')\,\mathrm{Im}(\varphi(re^{i\theta'})) > \alpha(\theta)\,\mathrm{Im}(\varphi(re^{i\theta'})) = \cot\tilde{\theta}\,\mathrm{Im}(\varphi(re^{i\theta'})).$$

On the other hand, if $\theta' \in (0, \pi/2]$ then, by our assumption,

$$\mathrm{Re}(\varphi(re^{i\theta'})) \ge \cot\gamma\, \mathrm{Im}(\varphi(re^{i\theta'})) > \alpha(\theta)\, \mathrm{Im}(\varphi(re^{i\theta'})) = \cot\tilde{\theta}\, \mathrm{Im}(\varphi(re^{i\theta'})).$$

Thus,

$$\varphi(\Sigma_\theta^+) \subset \Sigma_{\tilde{\theta}}^+,$$

and, in view of $\varphi(re^{-i\theta}) = \overline{\varphi(re^{i\theta})}$, the proposition follows. □

Let $\psi \in \mathcal{BF}$. By Theorem 2.4, (ii), we have $h(\lambda) := \lambda^2\widehat{\psi}(\lambda) \in \mathcal{CBF}$. From Theorem 2.4, (iii), it follows that $\varphi(\lambda) := \lambda h(1/\lambda) \in \mathcal{CBF}$, and

$$\varphi(\lambda) = \lambda^{-1}\widehat{\psi}(1/\lambda). \tag{2.17}$$

We will say that $\varphi \in \mathcal{CBF}$ given by (2.17) is *associated with* $\psi \in \mathcal{BF}$.

The notion of associated complete Bernstein function will be of primary importance in this paper, and we will first collect its several properties in Lemma 2.7 below. To this aim, the next auxiliary statement will be useful.

Lemma 2.6. *Define*

$$\Delta(\lambda) := \frac{1}{1+\lambda} - e^{-\lambda}, \qquad \lambda \in \overline{\mathbb{C}}_+. \tag{2.18}$$

Then

$$|\Delta(\lambda)| \le \frac{4\,|\lambda|^2}{(1+\mathrm{Re}\,\lambda)^3}, \qquad \lambda \in \overline{\mathbb{C}}_+. \tag{2.19}$$

Proof. Observe that

$$\Delta(\lambda) = \lambda^2 \int_0^1 e^{-\lambda s}(s-1+e^{-s})\,ds + \lambda^2 \int_1^\infty e^{-(\lambda+1)s}\,ds, \qquad \lambda \in \overline{\mathbb{C}}_+.$$

Since

$$s - 1 + e^{-s} \le \frac{s^2}{2}, \qquad e^{-s} \le \frac{2}{(s+1)^2}, \qquad s > 0,$$

we then have

$$|\lambda|^{-2}|\Delta(\lambda)| \le \frac{e}{2}\int_0^1 e^{-s(\mathrm{Re}\,\lambda+1)} s^2\,ds + \frac{e^{-1-\mathrm{Re}\,\lambda}}{1+\mathrm{Re}\,\lambda} \le \frac{4}{(1+\mathrm{Re}\,\lambda)^3}$$

for $\lambda \in \mathbb{C}_+$. □

Lemma 2.7. *Let $\varphi \in \mathcal{CBF}$ be associated with $\psi \in \mathcal{BF}$ and let $\psi \sim (a, b, \mu)$. Then*

a) *φ admits the representation*

$$\varphi(\lambda) = a + b\lambda + \int_{0+}^{\infty} \frac{\lambda s\, \mu(ds)}{1 + \lambda s}, \qquad \lambda \in \mathbb{C} \setminus (-\infty, 0];$$

b) *one has*

$$\mathrm{Re}(\psi(\lambda)) \geq \varphi(\mathrm{Re}\,\lambda), \qquad \lambda \in \mathbb{C}_+;$$

c) *one has*

$$|\psi(\lambda) - \varphi(\lambda)| \leq 2|\lambda|^2 |\varphi''(\mathrm{Re}\,\lambda)|, \qquad \lambda \in \mathbb{C}_+, \tag{2.20}$$

and, moreover, for every $\beta \in (0, \pi/2)$,

$$|\psi(\lambda) - \varphi(\lambda)| \leq \frac{4|\lambda|}{\cos\beta}\, \varphi'(\mathrm{Re}\,\lambda), \qquad \lambda \in \overline{\Sigma}_\beta \setminus \{0\}; \tag{2.21}$$

d) *ψ is bounded on $\mathbb{R}_+$ if and only if φ is bounded on $\mathbb{R}_+$. Moreover, if either ψ or φ is bounded on $\mathbb{R}_+$, then for any $\beta \in (0, \pi/2)$, the limits $\lim_{\lambda\to\infty, \lambda\in\Sigma_\beta} \psi(\lambda)$ and $\lim_{\lambda\to\infty, \lambda\in\Sigma_\beta} \varphi(\lambda)$ exist and are equal.*

Proof. The assertion a) follows directly from (2.17) and (2.2). To prove b) we note that

$$1 - e^{-\tau} \geq \frac{\tau}{1 + \tau}, \quad \tau > 0.$$

Then, setting $u = \mathrm{Re}\,\lambda > 0$, from Proposition 2.3, (ii) and a), it follows that

$$\begin{aligned} \mathrm{Re}(\psi(\lambda)) \geq \psi(u) &= a + bu + \int_{0+}^{\infty} (1 - e^{-us})\, \mu(ds) \\ &\geq a + bu + \int_{0+}^{\infty} \frac{us\, \mu(ds)}{1 + us} \\ &= \varphi(u), \end{aligned}$$

so that b) holds.

Let us now prove c). Observe that by Lemma 2.6 and a),

$$\psi(\lambda) - \varphi(\lambda) = \int_{0+}^{\infty} \Delta(\lambda s)\, \mu(ds), \quad \lambda \in \mathbb{C}_+, \tag{2.22}$$

where Δ is defined by (2.18). By a),

$$\varphi''(\lambda) = -2\int_{0+}^{\infty} \frac{s^2\,\mu(ds)}{(1+\lambda s)^3}, \qquad \lambda \in \mathbb{C}_+.$$

Then, using Lemma 2.6 and (2.22), it follows that

$$|\psi(\lambda) - \varphi(\lambda)| \le \int_{0+}^{\infty} |\Delta(\lambda s)|\,\mu(ds) \le 4|\lambda|^2 \int_{0+}^{\infty} \frac{s^2\,\mu(ds)}{(1+us)^3} = 2|\lambda|^2|\varphi''(u)|,$$

so that (2.20) holds.

To show (2.21), we note that $\varphi \in \mathcal{CBF}$ implies $|s\varphi''(s)| \le 2\varphi'(s)$, $s > 0$, see [16, p. 205]. Using (2.20) and observing that $\cos\beta|\lambda| \le \operatorname{Re}\lambda$, $\lambda \in \Sigma_\beta$, we obtain (2.21).

To prove the first statement in d), it suffices to note that if either ψ or φ is bounded then $b = 0$ and the measure μ is bounded by Fatou's theorem, see [32, Proposition 3.8, (v)]. Moreover, since $|\Delta(\lambda)| \le 2$, $\lambda \in \mathbb{C}_+$, and

$$\lim_{\lambda\to\infty,\lambda\in\Sigma_\beta} |\Delta(\lambda)| = 0,$$

for any $\beta \in (0, \pi/2)$, the equality (2.22) implies the second assertion in d) by the bounded convergence theorem. □

Now we are ready to prove the main result of this section providing an estimate for the difference of "resolvents" of a Bernstein function and the complete Bernstein function associated to it.

Theorem 2.8. *Let $\varphi \in \mathcal{CBF}$ be associated with $\psi \in \mathcal{BF}$. Let $\omega \in (\pi/2, \pi)$ and $z \in \Sigma_\omega$ be fixed. If*

$$r(\lambda; z) := \frac{1}{z + \psi(\lambda)} - \frac{1}{z + \varphi(\lambda)}, \qquad \lambda \in \Sigma_{\pi-\omega},$$

then the function $r(\cdot; z)$ is holomorphic in $\Sigma_{\pi-\omega}$ and for every $\beta \in (0, \pi-\omega)$:

$$\int_{\partial\Sigma_\beta} |r(\lambda; z)| \frac{|d\lambda|}{|\lambda|} \le \frac{8}{\cos^2\beta\,\cos^2((\omega+\beta)/2)\,|z|}.$$

Proof. Note first that $\pi - \omega \in (0, \pi/2)$. Since by Proposition 2.2, the functions ψ and φ preserve sectors, $z + \psi$ and $z + \varphi$ are not zero at each point from $\Sigma_{\pi-\omega}$. As ψ and φ are holomorphic in $\mathbb{C}_+$, the holomorphy of $r(\cdot, z)$ in $\Sigma_{\pi-\omega}$ follows.

Let now $\beta \in (0, \pi - \omega)$ and $0 \ne \lambda \in \Sigma_\beta$, $z \in \Sigma_\omega$. If $K = \cos((\omega + \beta)/2)$, then by Proposition 2.3, (i), we have

$$K^2|r(\lambda;z)| \le \frac{|\varphi(\lambda)-\psi(\lambda)|}{(|z|+|\psi(\lambda)|)(|z|+|\varphi(\lambda)|)}. \tag{2.23}$$

Let us estimate the numerator and the denominator in the right hand side of (2.23) separately. By (2.21),

$$|\varphi(\lambda)-\psi(\lambda)| \le \frac{4|\lambda|}{\cos\beta}\varphi'(\operatorname{Re}\lambda),$$

and, moreover, Proposition 2.3, (ii) and Lemma 2.7, b) yield

$$\begin{aligned}(|z|+|\psi(\lambda)|)(|z|+|\varphi(\lambda)|) &\ge (|z|+\operatorname{Re}(\psi(\lambda)))(|z|+\operatorname{Re}(\varphi(\lambda)))\\ &\ge (|z|+\varphi(\operatorname{Re}\lambda))^2.\end{aligned}$$

Thus, if $\lambda = te^{\pm i\beta}$, $t>0$, then

$$K^2|r(\lambda;z)| \le \frac{4t}{\cos\beta}\frac{\varphi'(t\cos\beta)}{(|z|+\varphi(t\cos\beta))^2}. \tag{2.24}$$

Hence,

$$\begin{aligned}K^2\int_{\partial\Sigma_\beta}|r(\lambda;z)|\frac{|d\lambda|}{|\lambda|} &\le \frac{8}{\cos\beta}\int_0^\infty \frac{\varphi'(t\cos\beta)\,dt}{(|z|+\varphi(t\cos\beta))^2}\\ &\le \frac{8}{\cos^2\beta\,|z|},\end{aligned}$$

and Theorem 2.8 follows. □

Corollary 2.9. *If $r(\lambda;z)$ is defined as in Theorem 2.8, then for all $z\in\Sigma_\omega$ and $\lambda\in\Sigma_\beta$:*

$$r(\lambda;z) = \frac{1}{2\pi i}\int_{\partial\Sigma_\beta}\frac{r(\mu;z)\,d\mu}{\mu-\lambda}, \tag{2.25}$$

and

$$\int_{\partial\Sigma_\beta}\frac{\lambda^k r(\lambda;z)}{(\lambda+1)^2}\,d\lambda = 0, \qquad k=0,1, \tag{2.26}$$

where the contour $\partial\Sigma_\beta$ is oriented counterclockwise.

Proof. If ψ is unbounded on $\mathbb{R}_+$ then φ is unbounded on $\mathbb{R}_+$ as well by Lemma 2.7, d), so using (2.24) and (2.6) we obtain that

$$|r(\lambda;z)| = \mathrm{o}(1) \quad \text{uniformly in} \quad \lambda\in\Sigma_\beta, \quad \lambda\to\infty, \tag{2.27}$$

for any $z \in \Sigma_\omega$. If ψ is bounded on $\mathbb{R}_+$ then (2.27) follows directly from (2.23) and Lemma 2.7, d). Now (2.27), Theorem 2.8 and a standard argument based on Cauchy's integral formula yield the representation (2.25). Finally, (2.26) is a consequence of (2.27) and Theorem 2.8. □

We finish this section with formulating Bochner's theorem which is at the heart of the notion of subordination. Recall that a family of positive Borel measures $(\mu_t)_{t\geq 0}$ on $[0,\infty)$ is called a vaguely continuous convolution semigroup of (subprobability) measures if for all $t, s \geq 0$,

$$\mu_t([0,\infty)) \leq 1, \qquad \mu_{t+s} = \mu_t * \mu_s, \qquad \text{and} \qquad \text{vague} - \lim_{t\to 0+} \mu_t = \delta_0,$$

where δ_0 stands for the Dirac measure at zero, and $*$ denotes convolution. Note that necessarily $\mu_0 = \delta_0$. The following classical result due to Bochner can be found e.g. in [32, Theorem 5.2].

Theorem 2.10. *The function $\psi : (0,\infty) \to [0,\infty)$ is Bernstein if and only if there exists a (unique) vaguely continuous convolution semigroup of subprobability measures $(\mu_t)_{t\geq 0}$ on $[0,\infty)$ such that*

$$\widehat{\mu}_t(\lambda) = \int_0^\infty e^{-\lambda s}\, \mu_t(ds) = e^{-t\psi(\lambda)}, \qquad \lambda > 0, \tag{2.28}$$

for all $t \geq 0$.

3. Preliminaries on functional calculi

In this section, we present basics on the functional calculi theory important for the sequel. We also prove several auxiliary results on continuity and compatibility of functional calculi.

3.1. Sectorial operators and holomorphic functional calculus

There are several ways to define a function of a sectorial operator. Probably the most well-known approach to that task is provided by the holomorphic functional calculus. We set up this calculus below omitting some crucial details and referring to [13, Sections 1–2] and [20] for more information.

A closed, densely defined linear operator A on X is called *sectorial of angle* $\omega \in [0,\pi)$ if $\mathbb{C} \setminus \overline{\Sigma}_\omega \subset \rho(A)$ and

$$M(A,\omega') := \sup\{\|\lambda(\lambda - A)^{-1}\| : \lambda \notin \overline{\Sigma}_{\omega'}\} < \infty$$

for all $\omega' \in (\omega,\pi)$. The set of sectorial operators of angle ω will be denoted by $\mathrm{Sect}(\omega)$.

It is well-known that A is sectorial if and only if $(-\infty, 0) \subset \rho(A)$ and

$$M(A) := \sup_{s>0} \|s(s+A)^{-1}\| < \infty, \tag{3.1}$$

see e.g. [13, Proposition 2.1.1, a)].

For $\theta \in (0, \pi)$, let

$$H_0^\infty(\Sigma_\theta) := \left\{ f \in \mathcal{O}(\Sigma_\theta) : |f(\lambda)| \le C \min(|\lambda|^\alpha, |\lambda|^{-\alpha}) \text{ for some } C, \alpha > 0 \right\},$$

where $\mathcal{O}(\Sigma_\theta)$ denotes the algebra of all holomorphic functions in Σ_θ. Let also

$$\mathcal{B}(\Sigma_\theta) = \left\{ f \in \mathcal{O}(\Sigma_\theta) : |f(\lambda)| \le C \max\left(|\lambda|^\alpha, |\lambda|^{-\alpha}\right) \text{ for some } C, \alpha > 0 \right\}.$$

Let $A \in \mathrm{Sect}(\omega)$, and let $\omega < \theta < \pi$. For $f \in H_0^\infty(\Sigma_\theta)$, define

$$\Phi(f) = f(A) := \frac{1}{2\pi i} \int_{\partial\Sigma_{\omega'}} f(\lambda)(\lambda - A)^{-1}\, d\lambda, \tag{3.2}$$

where $\partial\Sigma_{\omega'}$ is the downward oriented boundary of a sector $\Sigma_{\omega'}$, with $\omega' \in (\omega, \theta)$. This definition is independent of ω'. The mapping Φ is said to be *the holomorphic functional calculus* for A.

The holomorphic functional calculus Φ can be extended to a larger class of holomorphic functions. Given $f \in \mathcal{B}(\Sigma_\theta)$, let $e \in H_0^\infty(\Sigma_\theta)$ be such that $ef \in H_0^\infty(\Sigma_\theta)$ and the operator $\Phi(e) = e(A)$ is injective. Then we define

$$\Phi_e(f) = f(A) := [e(A)]^{-1}(ef)(A), \tag{3.3}$$

with the natural domain. The mapping Φ_e is called *the extended holomorphic functional calculus* for A. Such a definition does not depend on the choice of the function e called *regulariser*, and, moreover, $f(A)$ is a closed operator in X. Note that if A is injective and $\tau(\lambda) := \frac{\lambda}{(1+\lambda)^2}$, then $\tau^n(A)$ is injective for every $n \in \mathbb{N}$. In what follows, we will be sometimes considering injective A. In this case, one may put $e = \tau^n$ for large enough n.

Note that after an appropriate identification we may consider our extended calculus to be defined on the algebra

$$\mathcal{B}[\Sigma_\theta] := \bigcup_{\theta<\gamma<\pi} \mathcal{B}(\Sigma_\gamma).$$

We will frequently use the following properties of the extended holomorphic functional calculus. For their proofs see e.g. [13, Theorem 1.3.2], [13, Theorem 3.1.2] and [13, Proposition 3.1.4], respectively.

Proposition 3.1. *Let $A \in \mathrm{Sect}(\omega)$.*

(i) *The sum and product rules: If $f, g \in \mathcal{B}[\Sigma_\omega]$, then*

$$f(A) + g(A) \subset (f+g)(A) \qquad \textit{and} \qquad f(A)g(A) \subset (fg)(A). \tag{3.4}$$

Moreover, one has equality in the above relations if $g(A) \in \mathcal{L}(X)$.

(ii) *If $q > 0$, and $q\omega < \pi$, then $A^q \in \mathrm{Sect}(q\omega)$.*

(iii) *The composition rule: Let $q \in (0,1)$, and $f \in \mathcal{B}[\Sigma_{q\beta}]$. If A is injective, then $f(A^q) = (f \circ \lambda^q)(A)$.*

3.2. Hirsch functional calculus

We now define complete Bernstein functions of sectorial operators following Hirsch and review some of their basic properties needed in the sequel. Let A be a sectorial operator on X. The next definition was essentially given in [15, p. 255], see also [2].

Given $f \in \mathcal{CBF}$ with Stieltjes representation (a, b, σ) (see (2.9)), define $f_0(A) : \mathrm{dom}(A) \to X$ by

$$f_0(A)x = ax + bAx + \int_{0+}^{\infty} A(s+A)^{-1}x\,\sigma(ds), \qquad x \in \mathrm{dom}(A).$$

By (2.10) and (3.1), the integral above is absolutely convergent and $f_0(A)(1+A)^{-1}$ is a bounded operator on X, extending $(1+A)^{-1}f_0(A)$. Hence $f_0(A)$ is closable as an operator on X. Put

$$f(A) = \overline{f_0(A)}. \tag{3.5}$$

The operator $f(A)$ is called a *complete Bernstein function of* A. Note that by the above definition, $\mathrm{dom}(A)$ is core for $f(A)$.

The mapping $f \mapsto f(A)$ given by (3.5) is called the *Hirsch functional calculus* for A. Several properties of this calculus are described below. Their proofs can be found in e.g. [15, Théorème 1–3].

Theorem 3.2. *Let A be a sectorial operator on X, and let f and g be complete Bernstein functions. Then the following statements hold.*

(i) *The operator $f(A)$ is sectorial and*

$$\sup_{s>0} \|s(s+f(A))^{-1}\| \le \sup_{s>0} \|s(s+A)^{-1}\|.$$

(ii) *The composition rule:* $f(g(A)) = (f \circ g)(A)$. *In particular, for* $q \in (0,1)$ *define* $f_q(\lambda) := f(\lambda^q)$ *and* $g_{1/q}(\lambda) := [g(\lambda)]^{1/q}$. *Then*

$$f_q(A) = f(A^q), \qquad \textit{and} \qquad [g_{1/q}(A)]^q = g(A) \quad \textit{if} \quad g_{1/q} \in \mathcal{CBF}.$$

Note that, in contrast to Proposition 3.1, Theorem 3.2 does not require the injectivity of A. However, Theorem 3.2 deals only with complete Bernstein functions, while Proposition 3.1 holds for a larger class of functions.

3.3. Hille–Phillips functional calculus

Let $\mathrm{M_b}(\mathbb{R}_+)$ be the Banach algebra of bounded Borel measures on $\mathbb{R}_+$. If

$$\mathrm{A}^1_+(\mathbb{C}_+) := \{\widehat{\mu} : \mu \in \mathrm{M}_b(\mathbb{R}_+)\}$$

then $\mathrm{A}^1_+(\mathbb{C}_+)$ is a commutative Banach algebra with pointwise multiplication and with the norm

$$\|\hat{\mu}\|_{\mathrm{A}^1_+(\mathbb{C}_+)} := \|\mu\|_{\mathrm{M_b}(\mathbb{R}_+)} = |\mu|(\mathbb{R}_+),$$

where $|\mu|(\mathbb{R}_+)$ stands for the total variation of μ on $\mathbb{R}_+$.

Let $(e^{-tA})_{t\geq 0}$ be a bounded C_0-semigroup on X. Then the mapping $\Phi : \mathrm{A}^1_+(\mathbb{C}_+) \mapsto \mathcal{L}(X)$ defined by

$$\Phi(\widehat{\mu})x := \int_0^\infty e^{-sA}x\,\mu(ds), \qquad x \in X,$$

is a continuous algebras homomorphism. The homomorphism Φ is called the *Hille–Phillips* (*HP*-) functional calculus for A, and one sets

$$\widehat{\mu}(A) = \Phi(\widehat{\mu}).$$

Basic properties of the Hille–Phillips functional calculus can be found in [14, Chapter XV].

As in the case of holomorphic functional calculus, the *HP*-calculus for A can be extended to a larger class of functions. We will need a version of that extension procedure suitable to our purposes. Setting $e(\lambda) := 1/(\lambda+1) \in \mathrm{A}^1_+(\mathbb{C}_+)$, let $\mathcal{A}$ be the set of f holomorphic in $\mathbb{C}_+$ such that $ef \in \mathrm{A}^1_+(\mathbb{C}_+)$. Then $\mathcal{A}$ is an algebra. For $f \in \mathcal{A}$ define a closed linear operator $f(A)$ similarly to (3.3):

$$\Phi_e(f) = f(A) := (1+A)[f(\lambda)(1+\lambda)^{-1}](A). \tag{3.6}$$

The mapping Φ_e given by (3.6) will be called *the extended Hille–Phillips (HP-) calculus* for A. It is crucial to note that the extended *HP*-calculus satisfies the same sum and product rules as in Proposition 3.1, (i) up to replacement of the algebra $B[\Sigma_\omega]$ by the algebra $\mathcal{A}$.

Since, according to [11, Lemma 2.5], for any $\psi \in \mathcal{BF}$ one has $\psi(\lambda)/(1+\lambda) \in \mathrm{A}^1_+(\mathbb{C}_+)$, one can define a Bernstein function ψ of A by (3.6) in the extended *HP*-calculus, see [11] for more details.

It was proved in [11, Corollary 2.6] that the formula (3.6) can be written in the following more explicit and useful form which is an operator analogue of the Lévy–Hintchine representation of ψ.

Proposition 3.3. *Let $-A$ be the generator of a bounded C_0-semigroup $(e^{-tA})_{t\ge 0}$ on X, and let $\psi \in \mathcal{BF}$, $\psi \sim (a, b, \mu)$. If $\psi(A)$ is defined by (3.6), then*

$$\psi(A)x = ax + bAx + \int_{0+}^{\infty} (1 - e^{-sA})x\, \mu(ds), \qquad x \in \operatorname{dom}(A), \tag{3.7}$$

where the integral is understood as a Bochner integral, and $\operatorname{dom}(A)$ *is core for* $\psi(A)$.

In other approaches to defining $\psi(A)$ (see e.g. [27,16], [32, Chapter 13]), the representation (3.7) of $\psi(A)$ holds as well. Thus, (3.7) allows one to use the standard results on operator Bernstein functions in the framework of *the extended HP-calculus.* In particular, in view of [32, Proposition 13.1 and Theorem 13.6], for any $\psi \in \mathcal{BF}$ the operator $-\psi(A)$ defined by (3.6) generates a bounded C_0-semigroup $(e^{-t\psi(A)})_{t\ge 0}$ on X, and $(e^{-t\psi(A)})_{t\ge 0}$ can be represented in terms of ψ and $(e^{-tA})_{t\ge 0}$ as (2.28) suggests:

$$e^{-t\psi(A)} = \int_{0}^{\infty} e^{-sA}\, \mu_t(ds), \qquad t \ge 0, \tag{3.8}$$

where $(\mu_t)_{t\ge 0}$ is a vaguely continuous convolution semigroup of subprobability measures on $[0, \infty)$ corresponding to ψ via (2.28).

The semigroup $(e^{-t\psi(A)})_{t\ge 0}$ is called *subordinate* to the semigroup $(e^{-tA})_{t\ge 0}$ with respect to the Bernstein function ψ. The relation (3.8) implies that $(-\infty, 0) \subset \rho(\psi(A))$ and

$$\sup_{s>0} \|s(s + \psi(A))^{-1}\| \le \sup_{t>0} \|e^{-t\psi(A)}\| \le \sup_{t>0} \|e^{-tA}\|. \tag{3.9}$$

From Proposition 3.3 it follows that the composition rule for Bernstein functions of semigroup generators proved in [32, Theorem 13.23, (iii)] holds in the setting of the extended *HP*-calculus too: if $\psi, \varphi \in \mathcal{BF}$ and $-A$ is the generator of a bounded C_0-semigroup, then

$$(\psi \circ \varphi)(A) = \psi(\varphi(A)). \tag{3.10}$$

This is a version of the composition rule in Theorem 3.2, (ii).

The next approximation result will often be useful.

Proposition 3.4. *Let $-A$ be the generator of a bounded C_0-semigroup on X and let $\psi \in \mathcal{BF}$.*

(i) *If $\epsilon > 0$ then $[\psi(\cdot + \epsilon) - \psi(\cdot)](A) \in \mathcal{L}(X)$ so that $\mathrm{dom}(\psi(A+\epsilon)) = \mathrm{dom}(\psi(A))$ and*

$$\psi(A+\epsilon) = \psi(A) + [\psi(\cdot + \epsilon) - \psi(\cdot)](A).$$

(ii) *If $x \in \mathrm{dom}(A)$ and $\epsilon > 0$, then*

$$\|\psi(A+\epsilon)x - \psi(A)x\| \le M(\psi(\epsilon) - \psi(0))\|x\|,$$

where $M := \sup_{t\ge 0} \|e^{-tA}\|$.

(iii) *For all $s > 0$ and $x \in X$,*

$$\lim_{\epsilon\to 0+} \|(s + \psi(A+\epsilon))^{-1}x - (s+\psi(A))^{-1}x\| = 0.$$

Proof. To prove (i) and (ii), we note that if ψ is a Bernstein function with the Lévy–Hintchine representation (a, b, μ) then for all $\epsilon > 0$ and $\lambda \in \mathbb{C}_+$,

$$\psi(\lambda + \epsilon) - \psi(\lambda) = b\epsilon + \int_{0+}^{\infty} e^{-\lambda s}(1 - e^{-\epsilon s})\,\mu(ds).$$

Hence, $\psi(\cdot + \epsilon) - \psi(\cdot) \in A^1_+(\mathbb{C}_+)$ and

$$\|\psi(\cdot + \epsilon) - \psi(\cdot)\|_{A^1_+} = \psi(\epsilon) - \psi(0).$$

From here by the *HP*-calculus it follows that

$$\|[\psi(\cdot + \epsilon) - \psi(\cdot)](A)\| \le M(\psi(\epsilon) - \psi(0)).$$

Moreover, since $[\psi(\cdot+\epsilon) - \psi(\cdot)](A) \in \mathcal{L}(X)$, by the sum rule for the extended *HP*-calculus we obtain that $\mathrm{dom}(\psi(A+\epsilon)) = \mathrm{dom}(\psi(A))$ and (i) holds. Since $\mathrm{dom}(A) \subset \mathrm{dom}(\psi(A))$, by Proposition 3.3, we have also

$$\|\psi(A+\epsilon)x - \psi(A)x\| \le M[\psi(\epsilon) - \psi(0)]\|x\|$$

for all $x \in \mathrm{dom}(A)$, i.e. (ii) is true.

Furthermore, by the product rule for the extended *HP*-calculus, for all $\epsilon, s > 0$ and $x \in \mathrm{dom}(A)$,

$$(s + \psi(A))^{-1}(\psi(A+\epsilon) - \psi(A))x = (\psi(A+\epsilon) - \psi(A))(s + \psi(A))^{-1}x.$$

Hence, using (ii) and the estimate (3.9), we obtain:

$$\begin{aligned}
&\|(s+\psi(A))^{-1}x - (s+\psi(A+\epsilon))^{-1}x\| \\
&\quad = \|(s+\psi(A))^{-1}(s+\psi(A+\epsilon))^{-1}(\psi(A+\epsilon)-\psi(A))x\| \\
&\quad \le \frac{M^3}{s^2}(\psi(\epsilon)-\psi(0))\|x\| \\
&\quad \to 0, \quad \epsilon \to 0+, \quad x \in \mathrm{dom}(A).
\end{aligned}$$

As $\mathrm{dom}(A)$ is dense in X, (iii) follows. □

3.4. Compatibility of functional calculi

In this subsection we show that in several cases of interest the three functional calculi introduced above are compatible in a natural sense. This will allow us to employ these calculi simultaneously thus using specific relations and properties of each of them.

The first result shows that the extended holomorphic functional calculus and the Hirsch functional calculus are compatible.

Proposition 3.5. *Let $\varphi \in \mathcal{CBF}$ and let A be an injective sectorial operator on X. Then the operator $\varphi(A)$ defined by the Hirsch calculus coincides with $\varphi(A)$ defined via the extended holomorphic functional calculus. If $\varphi(\lambda) = \lambda^q, q \in (0,1)$, then the above holds for arbitrary sectorial A.*

Proposition 3.5 was proved in [2, Theorem 4.12] for injective A. The fact that it is true for $\varphi(\lambda) = \lambda^q, q \in (0,1)$, and *all* sectorial A can be proved by repeating the argument from [2] with the regulariser $\left(\lambda/(1+\lambda)^2\right)^n, n \in \mathbb{N}$, replaced by the regulariser $(1+\lambda)^{-1}$.

The next statement relates the extended Hille–Phillips calculus and the extended holomorphic calculus.

Proposition 3.6. *Let $\psi \in \mathcal{BF}$. Suppose that ψ admits a holomorphic extension $\tilde{\psi}$ to Σ_ω for some $\omega \in (\pi/2, \pi)$ so that $\tilde{\psi} \in \mathcal{B}(\Sigma_\omega)$. Let $-A$ be the generator of a bounded C_0-semigroup and let A be injective. Let $\psi(A)$ be defined by the extended HP-calculus and $\tilde{\psi}(A)$ be defined via the extended holomorphic functional calculus. Then*

$$\tilde{\psi}(A) = \psi(A).$$

Proof. Recall that there is $n \in \mathbb{N}$ such that $\tau^n(\lambda) = \left(\frac{\lambda}{(\lambda+1)^2}\right)^n$ is a regulariser for $\tilde{\psi}$ in the extended holomorphic functional calculus. Let $\tau^n(A)$ be given by that calculus. By [11, Lemma 2.5], τ^n is a regulariser for ψ in the extended *HP*-calculus too, and we denote by $\tau_h^n(A)$ the function τ^n evaluated at A by means of the *HP*-calculus. Then, by [13, Proposition 3.3.2] on compatibility of the holomorphic and Hille–Phillips calculi, $\tau_h^n(A) = \tau^n(A)$. By the same [13, Proposition 3.3.2], we have $(\tau^n\tilde{\psi})(A) = (\tau_h^n\psi)(A)$. Hence,

$$\tilde{\psi}(A) = (\tau^n(A))^{-1}(\tau^n\tilde{\psi})(A) = (\tau_h^n(A))^{-1}(\tau_h^n\psi)(A) = \psi(A). \qquad \square$$

Finally, the last result in this subsection yields compatibility of the extended *HP*-calculus and the Hirsch calculus for $\psi \in \mathcal{BF}$ of a special form.

Proposition 3.7. *Let $\psi \in \mathcal{BF}$. Suppose there exists $\theta \in (\pi/2, \pi)$ such that ψ admits a holomorphic extension $\tilde{\psi}$ to Σ_θ, and*

$$\tilde{\psi}(\Sigma_\theta^+) \subset \Sigma_{\pi/2}^+. \tag{3.11}$$

Let $\alpha := \frac{\theta}{\pi}$. Then the following hold.

(i) *If $\tilde{\psi}_\alpha(\lambda) := \tilde{\psi}(\lambda^\alpha)$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, then $\tilde{\psi}_\alpha \in \mathcal{CBF}$.*
(ii) *If $-A$ is the generator of a bounded C_0-semigroup, then $A^{1/\alpha} \in \mathrm{Sect}(\pi/(2\alpha))$ and*

$$\psi(A) = \tilde{\psi}_\alpha(A^{1/\alpha}), \tag{3.12}$$

where $\psi(A)$ is defined by the extended HP-calculus, and $\tilde{\psi}_\alpha(A^{1/\alpha})$ is given by the Hirsch calculus.

Proof. Note that the function $\tilde{\psi}_\alpha$ is positive on $(0, \infty)$. Moreover, it extends continuously to zero and maps the upper half-plane H^+ into itself. Hence, using Theorem 2.4, (iii), we infer that $\tilde{\psi}_\alpha \in \mathcal{CBF}$, i.e. (i) holds.

First, let A be injective. Since $\tilde{\psi}_\alpha \in \mathcal{CBF}$, one has $\tilde{\psi} \in \mathcal{B}[\Sigma_\theta]$, so that $\tilde{\psi}_\alpha(A^{1/\alpha})$ and $\tilde{\psi}(A)$ are well defined in the extended holomorphic functional calculus. From Proposition 3.1, (iii) it follows that

$$\tilde{\psi}_\alpha(A^{1/\alpha}) = \tilde{\psi}(A).$$

Moreover, using Propositions 3.5 and 3.6, we infer that

$$\psi_\alpha(A^{1/\alpha}) = \tilde{\psi}_\alpha(A^{1/\alpha}) \quad \text{and} \quad \psi(A) = \tilde{\psi}(A),$$

where $\psi_\alpha(A^{1/\alpha})$ is given by the Hirsch calculus, and $\psi(A)$ is defined by the extended *HP*-calculus, so that (3.12) holds.

If A is not injective, then consider the family $A_\epsilon := A + \epsilon$, $\epsilon > 0$. By the above,

$$\psi(A_\epsilon) = \tilde{\psi}_\alpha(A_\epsilon^{1/\alpha}), \qquad \epsilon > 0.$$

Due to Proposition 3.4, (ii),

$$\lim_{\epsilon\to 0} \psi(A_\epsilon)x = \psi(A)x, \qquad x \in \mathrm{dom}(A).$$

On the other hand, by [13, Proposition 3.1.3],

$$\lim_{\epsilon\to 0} A_\epsilon^{1/\alpha}x = A^{1/\alpha}x, \qquad x \in \mathrm{dom}(A^{1/\alpha}),$$

hence, in view of [3, Proposition 5.16],

$$\lim_{\epsilon\to 0} \psi_\alpha(A_\epsilon^{1/\alpha})x = \psi_\alpha(A^{1/\alpha})x, \qquad x \in \mathrm{dom}(A^{1/\alpha}).$$

So,

$$\psi(A)x = \psi_\alpha(A^{1/\alpha})x, \qquad x \in \mathrm{dom}(A^{1/\alpha}).$$

Since $\mathrm{dom}(A^{1/\alpha})$ is core for $\psi(A)$ and $\psi_\alpha(A^{1/\alpha})$ by Proposition 3.3 and [11, Lemma 2.1], respectively, we obtain (3.12). □

4. Main results: holomorphy and preservation of angles

Recall that a C_0-semigroup $(e^{-tA})_{t\ge 0}$ on X is said to be holomorphic if it extends holomorphically to a sector Σ_θ for some $\theta \in (0, \frac{\pi}{2}]$ and the extension is bounded on $\Sigma_{\theta'} \cap \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$ for any $\theta' \in (0, \theta)$. In this case, we write $-A \in \mathcal{H}(\theta)$. If the extension is bounded in Σ'_θ whenever $0 < \theta' < \theta$, then $(e^{-tA})_{t\ge 0}$ is said to be a *sectorially bounded* holomorphic semigroup of angle θ, and we then write $-A \in \mathcal{BH}(\theta)$. (The word "sectorially" in the definition above is usually omitted in the relevant literature, as, for instance, in [1], [8] or [3].) Note that $(e^{-tA})_{t>0}$ may admit a holomorphic extension to Σ_θ as above without being sectorially bounded (as already one-dimensional examples show). A sectorially bounded holomorphic C_0-semigroup can be characterized in terms of the sectoriality property for its generator. Recall that if $\theta \in (0, \pi/2]$ then $-A \in \mathcal{BH}(\theta)$ if and only if $A \in \mathrm{Sect}(\pi/2 - \theta)$, see e.g. [1, Theorem 3.7.11].

Berg, Boyadzhiev and de Laubenfels proved in [3, Propositions 7.1 and 7.4] that if $-A \in \mathcal{BH}(\theta)$ and $\theta \in (\pi/4, \pi/2]$, then for any $\psi \in \mathcal{BF}$ the operator $-\psi(A)$ generates a sectorially bounded holomorphic C_0-semigroup, and if $-A \in \mathcal{BH}(\pi/2)$, then $-\psi(A) \in \mathcal{BH}(\pi/2)$ too. They also asked in [3] whether the statement holds for θ from the whole of the interval $(0, \pi/2]$. In Theorem 4.5 below, we remove the restriction on θ and prove the result in full generality thus solving the problem posed in [3]. Moreover, we show that

$(e^{-t\psi(A)})_{t\ge 0}$ is holomorphic in the holomorphy sector of $(e^{-tA})_{t\ge 0}$. As byproduct, in Corollary 4.7, we also answer the question by Kishimoto–Robinson from [18] mentioned in Introduction. To this aim, we will first need to prove several results on functional calculi allowing one to apply Theorem 2.8.

First, we will need an estimate for the resolvent of $\varphi(A)$. In [3, Theorem 6.1 and Remark 6.2] it was proved that if $\varphi \in \mathcal{CBF}$ then

$$A \in \mathrm{Sect}(\omega) \Longrightarrow \varphi(A) \in \mathrm{Sect}(\omega), \qquad \omega \in [0, \pi/2). \tag{4.1}$$

The proof of (4.1) there was based on the fact that

$$\varphi \in \mathcal{CBF} \Longrightarrow [\varphi(\lambda^q)]^{1/q} \in \mathcal{CBF}, \qquad q \in (0,1), \tag{4.2}$$

and on a result similar to Theorem 3.2. In the statement below, we extend (4.1) to the whole class of sectorial operators.

Theorem 4.1. *Let $A \in \mathrm{Sect}(\omega)$, $\omega \in [0,\pi)$. If $\varphi \in \mathcal{CBF}$, then $\varphi(A) \in \mathrm{Sect}(\omega)$ too.*

Proof. In this proof, we will combine the (extended) holomorphic functional calculus and the Hirsch functional calculus. This is possible due to compatibility of the calculi given by Proposition 3.5.

Assume first that $\omega > 0$. Let $q \in (1, \pi/\omega)$. Recall first that by Theorem 3.2, (i), the operator $\varphi(A)$ is sectorial. For $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, define $g_{1/q}(\lambda) := \lambda^{1/q}$ and $g_q(\lambda) := \lambda^q$. Note that $\varphi_q := g_q \circ \varphi \circ g_{1/q} \in \mathcal{CBF}$ (see [32, Corollary 7.15]). Moreover, $\varphi \circ g_{1/q} \in \mathcal{CBF}$, and A^q is sectorial in view of Proposition 3.1, (ii). Now we use the Hirsch functional calculus. By Theorem 3.2, (i) the operator $[\varphi \circ g_{1/q}](A^q)$ is sectorial, so by Theorem 3.2, (ii) we infer that

$$[\varphi_q(A^q)]^{1/q} = [(g_q \circ \varphi \circ g_{1/q})(A^q)]^{1/q} = [\varphi \circ g_{1/q}](A^q).$$

Furthermore, observe that $(A^q)^{1/q} = A$ by the (extended) holomorphic functional calculus. Hence, taking into account Proposition 3.5 for power functions and using Theorem 3.2, (ii), we conclude that

$$[\varphi \circ g_{1/q}](A^q) = \varphi((A^q)^{1/q}) = \varphi(A).$$

Therefore

$$\varphi(A) = [\varphi_q(A^q)]^{1/q}, \tag{4.3}$$

and, due to Proposition 3.1, (ii), we obtain that

$$\varphi(A) \in \mathrm{Sect}(\pi/q). \tag{4.4}$$

Since $q \in (1, \pi/\omega)$ is arbitrary, it follows that $\varphi(A) \in \mathrm{Sect}(\omega)$.

If $\omega = 0$ then, since $\mathrm{Sect}(0) = \cap_{\omega\in(0,\pi)} \mathrm{Sect}(\omega)$, the above arguments yield $\varphi(A) \in \mathrm{Sect}(0)$. $\square$

Let now $-A \in \mathcal{BH}(\theta)$ for some $\theta \in (0, \pi/2]$ so that for every $\omega \in (0, \pi/2 + \theta)$,

$$\|(\lambda + A)^{-1}\| \le \frac{M(A,\omega)}{|\lambda|}, \qquad \lambda \in \Sigma_\omega. \tag{4.5}$$

Let $\omega \in (\pi/2, \pi/2+\theta)$ be fixed. Let also ψ be a Bernstein function, φ be the complete Bernstein function associated to ψ, and the function r be defined as in Theorem 2.8.

Define $r(A, \cdot) : \Sigma_\omega \to \mathcal{L}(X)$ and $F(A, \cdot) : \Sigma_\omega \to \mathcal{L}(X)$ by

$$r(A; z) := \frac{1}{2\pi i} \int_{\partial\Sigma_\beta} r(\lambda; z)(\lambda - A)^{-1}\, d\lambda, \tag{4.6}$$

and

$$F(A; z) := \frac{1}{2\pi i} \int_{\partial\Sigma_\beta} \frac{\lambda r(\lambda; z)}{(\lambda + 1)^2}(\lambda - A)^{-1}\, d\lambda, \tag{4.7}$$

where $\beta \in (\pi/2 - \theta, \pi - \omega)$ is arbitrary and Σ_β is oriented counterclockwise. In view of Theorem 2.8 and Cauchy's theorem, the functions r and F are well-defined.

We continue by providing sectoriality estimates for r in appropriate sectors and expressing F via r.

Proposition 4.2. *Let $-A \in \mathcal{BH}(\theta)$ for some $\theta \in (0, \pi/2]$ so that (4.5) holds. Then for every $\omega \in (\pi/2, \pi/2+\theta)$, $r(A, \cdot)$ is holomorphic in Σ_ω and for every $\beta \in (\pi/2-\theta, \pi-\omega)$,*

$$\|r(A; z)\| \le \frac{4M(A, \pi - \beta)}{\pi \cos^2 \beta\, \cos^2((\omega + \beta)/2)\, |z|}, \quad z \in \Sigma_\omega.$$

Proof. The estimate for $r(A; z)$ follows from (4.6), Theorem 2.8 and (4.5). The holomorphy of $r(A, \cdot)$ in Σ_ω is a direct consequence of Fubini's and Morera's theorems. $\square$

Lemma 4.3. *Let $r(A; z)$ and $F(A; z)$ be defined by (4.6) and (4.7), respectively. Then for every $\omega \in (\pi/2, \pi/2 + \theta)$,*

$$F(A; z) = A(A + 1)^{-2} r(A; z), \qquad z \in \Sigma_\omega.$$

Proof. Note that for every $\lambda \in \mathbb{C} \setminus (-\infty, 0)$,

$$\begin{aligned}\frac{\lambda}{(\lambda + 1)^2} - A(A + 1)^{-2} &= [\lambda(A + 1)^2 - (\lambda + 1)^2 A]\frac{(A + 1)^{-2}}{(\lambda + 1)^2}\\ &= (\lambda A - 1)(A - \lambda)\frac{(A + 1)^{-2}}{(\lambda + 1)^2}.\end{aligned}$$

Therefore, by (2.26), for every $x \in X$ one has

$$\begin{aligned} &F(A;z)x - A(A+1)^{-2}r(A;z)x \\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_\beta} r(\lambda;z)\left[\frac{\lambda}{(1+\lambda)^2} - A(A+1)^{-2}\right](\lambda - A)^{-1}x\,d\lambda \\ &= -\frac{1}{2\pi i}\int_{\partial\Sigma_\beta} \frac{r(\lambda;z)}{(\lambda+1)^2}(\lambda A - 1)(A+1)^{-2}x\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\partial\Sigma_\beta} \frac{r(\lambda;z)}{(\lambda+1)^2}(A+1)^{-2}x\,d\lambda \\ &\quad - \frac{1}{2\pi i}\int_{\partial\Sigma_\beta} \frac{\lambda r(\lambda;z)}{(\lambda+1)^2}A(A+1)^{-2}x\,d\lambda \\ &= 0. \qquad \square \end{aligned}$$

The following statement relating the resolvents of $\psi(A)$ and $\varphi(A)$ will be basic for proving the main result of this paper, Theorem 4.5. It shows that the resolvents do not differ much as far their behavior at infinity is concerned. Note that if $\psi \in \mathcal{BF}$ and $-A \in \mathcal{BH}(\theta)$, $\theta \in (0, \pi/2]$, then $\psi(A)$ given by the extended *HP*-calculus coincide with $\psi(A)$ defined by the extended holomorphic calculus. The proof of this fact is the same as that of Proposition 3.6.

Proposition 4.4. *Let $-A \in \mathcal{BH}(\theta)$ for some $\theta \in (0, \pi/2]$. Then for every $\omega \in (\pi/2, \pi/2 + \theta)$,*

$$(z + \psi(A))^{-1} = (z + \varphi(A))^{-1} + r(A;z), \qquad z \in \Sigma_\omega. \tag{4.8}$$

Proof. Suppose first that A has dense range. Then the operators $(z+\psi)^{-1}(A)$ and $(z+\varphi)^{-1}(A)$ are well-defined for $z > 0$ via the (extended) holomorphic functional calculus with the regulariser $\tau(\lambda) = \lambda/(1+\lambda)^2$. On the other hand, since $-\psi(A)$ and $-\varphi(A)$ generate bounded C_0-semigroups, we have that

$$(z + \psi(A))^{-1} \in \mathcal{L}(X) \quad \text{and} \quad (z + \varphi(A))^{-1} \in \mathcal{L}(X), \qquad z > 0.$$

Moreover, by [13, Theorem 1.3.2, *f*)], if $z > 0$, then

$$(z + \psi(A))^{-1} = (z+\psi)^{-1}(A) \quad \text{and} \quad (z + \varphi(A))^{-1} = (z+\varphi)^{-1}(A).$$

Hence, using the sum rule for the (extended) holomorphic functional calculus,

$$(z + \psi(A))^{-1} - (z + \varphi(A))^{-1} = [(z+\psi)^{-1} - (z+\varphi)^{-1}](A).$$

Furthermore, using the product rule for this calculus,

$$\begin{aligned}(z+\psi(A))^{-1}-(z+\varphi(A))^{-1} &= [\tau(A)]^{-1}[((z+\psi)^{-1}-(z+\varphi)^{-1})\tau](A)\\ &= [A(A+1)^{-2}]^{-1}[r(\cdot;z)\tau](A)\\ &= [A(A+1)^{-2}]^{-1}F(A;z).\end{aligned}$$

From the latter relation and Lemma 4.3 it follows that

$$(z+\psi(A))^{-1}=(z+\varphi(A))^{-1}+r(A;z), \qquad z>0. \tag{4.9}$$

To obtain (4.9) in case when the range of A may not be dense, we consider the approximation of A by the operators A_ϵ with dense range given by

$$A_\epsilon := A+\epsilon \in \mathcal{BH}(\theta), \quad \epsilon>0.$$

By (4.9) we have

$$(z+\psi(A_\epsilon))^{-1}-(z+\varphi(A_\epsilon))^{-1}=r(A_\epsilon;z), \qquad z>0, \quad \epsilon>0. \tag{4.10}$$

Next we use the extended *HP*-calculus. By applying Proposition 3.4, (iii) to the Bernstein functions ψ and φ, we infer that

$$\lim_{\epsilon\to 0}[(z+\psi(A_\epsilon))^{-1}-(z+\varphi(A_\epsilon))^{-1}]=(z+\psi(A))^{-1}-(z+\varphi(A))^{-1},$$

in the strong operator topology. On the other hand, since

$$|\lambda+\epsilon|\ge\cos(\beta/2)\,(|\lambda|+\epsilon), \qquad \lambda\in\partial\Sigma_\beta, \qquad \epsilon>0$$

(see (2.4)), for $\lambda\in\partial\Sigma_\beta$ one has

$$\|(A-\lambda)^{-1}-(A-\lambda-\epsilon)^{-1}\|\le\frac{\epsilon M^2(A,\pi-\beta)}{|\lambda(\lambda+\epsilon)|}\le\frac{\epsilon M^2(A,\pi-\beta)}{\cos(\beta/2)\,|\lambda|(|\lambda|+\epsilon)}.$$

So, by (4.6), Theorem 2.8 and the bounded convergence theorem, we obtain that

$$\begin{aligned}\|r(A;z)-r(A_\epsilon;z)\| &\le \frac{1}{2\pi}\int_{\partial\Sigma_\beta}|r(\lambda;z)|\|(A-\lambda)^{-1}-(A-\lambda-\epsilon)^{-1}\|\,|d\lambda|\\ &\le \frac{\epsilon M^2(A,\pi-\beta)}{2\pi}\int_{\partial\Sigma_\beta}\frac{|r(\lambda;z)|}{|\lambda|(|\lambda|+\epsilon)}\,|d\lambda|\\ &\to 0, \qquad \epsilon\to 0.\end{aligned}$$

Letting $\epsilon\to 0$ in (4.10), the equality (4.9) follows.

Thus, $(\cdot+\psi(A))^{-1}$ satisfies (4.9) and extends holomorphically to Σ_ω as both $r(\cdot, A)$ and $(\cdot+\varphi(A))^{-1}$ have the latter property by Proposition 4.2 and Theorem 4.1, respectively. Then, [1, Appendix B, Proposition B5] implies that $\Sigma_\omega \subset \rho(-\psi(A))$ and the extension is given by $(\cdot+\psi(A))^{-1}$. This yields (4.8) for all $z \in \Sigma_\omega$. □

Now we are ready to prove the main results of this paper. The first of them shows that Bernstein functions leave the class of generators of sectorially bounded holomorphic semigroups on a Banach space invariant and, moreover, preserve the holomorphy sectors.

Theorem 4.5. *Let $-A \in \mathcal{BH}(\theta)$ for some $\theta \in (0, \pi/2]$. Then for every $\psi \in \mathcal{BF}$ one has $-\psi(A) \in \mathcal{BH}(\theta)$.*

Proof. Let $\varphi \in \mathcal{CBF}$ be the function associated with $\psi \in \mathcal{BF}$. By Theorem 4.1, if $-A \in \mathcal{BH}(\theta)$ then $-\varphi(A) \in \mathcal{BH}(\theta)$. Taking into account Propositions 4.2 and 4.4, we infer that $\psi(A) \in \mathrm{Sect}(\pi-\omega)$ for every $\omega \in (\pi/2, \pi/2+\theta)$. Choosing ω arbitrarily close to $\pi/2+\theta$ we conclude that $\psi(A) \in \mathrm{Sect}(\pi/2-\theta)$. Hence, $-\psi(A) \in \mathcal{BH}(\theta)$. □

Theorem 4.5 has a version saying that Bernstein functions preserve the class of bounded (but not necessarily sectorially bounded) holomorphic C_0-semigroups. This version is an immediate consequence of Theorem 4.5 and the following lemma.

Lemma 4.6. *Let $-A$ be the generator of a bounded C_0-semigroup on X and let $\psi \in \mathcal{BF}$. Suppose there exists $d \ge 0$ such that $-\psi(A+d) \in \mathcal{H}(\theta)$ for some $\theta \in (0, \pi/2]$. Then $-\psi(A) \in \mathcal{H}(\theta)$.*

Proof. We use the extended *HP*-calculus. By Proposition 3.4, (i), we have

$$\psi(A+d) = \psi(A) + [\psi(\cdot+d) - \psi(\cdot)](A) = \psi(A) + B_d,$$

where $B_d \in \mathcal{L}(X)$. By the product rule for the (extended) *HP*-calculus we have

$$(\psi(A+d)+s)^{-1}(B_d+s)^{-1} = (B_d+s)^{-1}(\psi(A+d)+s)^{-1}$$

for sufficiently large $s > 0$. Then by [25, Section A-I.3.8, p. 24] the C_0-semigroups $(e^{-t\psi(A+d)})_{t\ge0}$ and $(e^{-tB_d})_{t\ge0}$ commute. Taking into account that $\mathrm{dom}(\psi(A)) = \mathrm{dom}(\psi(A+d))$ by Proposition 3.4, (ii) and using [8, Subsection II.2.7], we conclude that

$$e^{-t\psi(A)} = e^{-t\psi(A+d)}e^{tB_d}, \qquad t \ge 0.$$

Since $(e^{tB_d})_{t\ge0}$ extends to an entire function, the statement of the lemma follows. □

Corollary 4.7. *Let $-A$ be the generator of a bounded C_0-semigroup on X such that $-A \in \mathcal{H}(\theta)$ for some $\theta \in (0, \pi/2]$. Then for every $\psi \in \mathcal{BF}$ one has $-\psi(A) \in \mathcal{H}(\theta)$.*

Proof. Observe that if $(e^{-tA})_{t\geq 0}$ is a bounded C_0-semigroup admitting a holomorphic extension to Σ_θ, $\theta \in (0,\pi/2]$, then by e.g. [1, Proposition 3.7.2 *b*)] we infer that for fixed $\theta' \in (0,\theta)$ and big enough $d > 0$ the operator $-(d+A)$ generates a C_0-semigroup $(e^{-t(d+A)})_{t\geq 0}$ which is holomorphic and sectorially bounded in $\Sigma_{\theta'}$. Therefore, by Theorem 4.5 the C_0-semigroup $(e^{-t\psi(d+A)})_{t\geq 0}$ is also holomorphic and sectorially bounded in $\Sigma_{\theta'}$. By Lemma 4.6, $-\psi(A)$ generates a bounded C_0-semigroup which extends holomorphically to $\Sigma_{\theta'}$. Since the choice of $\theta' \in (0,\theta)$ is arbitrary, the corollary follows. □

Remark 4.8. It was claimed in [21, Theorem 7.1] that if $-A$ is the generator of a bounded C_0-semigroup on X then $-A \in \cup_{\theta\in(0,\pi/2]}\mathcal{H}(\theta)$ implies the same property for $-\psi(A)$. Unfortunately, the proof of this fact in [21] seems to contain a mistake. Specifically, in the notation of [21], the proof at its final stage relies on the boundedness of the operator $\psi(A)g_t(A)$ which was not proved in [21]. Nonetheless, the holomorphy of $(e^{-t\psi(A)})_{t\geq 0}$ was proved in [24, Theorem 13] for uniformly convex X by means of the Kato–Pazy criterion. (Concerning the Kato–Pazy criterion see [17] and [26, Corollaries 2.5.7 and 2.5.8].)

Let $-A$ be the generator of a bounded C_0-semigroup, and let the range $\operatorname{ran}(A)$ of A be dense (so that A is injective by the mean ergodic theorem, see e.g. [1, p. 261]). Consider so-called Stieltjes functions $f : (0,\infty) \to (0,\infty)$ which can be defined by the property that $1/f \in \mathcal{CBF}$. Note that for Stieltjes f the operator $-f(A)$ does not, in general, generate a C_0-semigroup. For example, if $f(z) = 1/z$ then $f(A) = A^{-1}$, and a counterexample can be found in [12]. On the other hand, for generators of sectorially bounded holomorphic C_0-semigroups the situation is different, as we prove in Corollary 4.10 below.

Let us first introduce the notion of potential and define several operators related to it. Recall (see e.g. [32, Definition 5.24]) that a function $f : (0,\infty) \mapsto (0,\infty)$ is said to be *potential*, if there exists $\psi \in \mathcal{BF}$ such that $f = 1/\psi$. The set of all potentials will be denoted by $\mathcal{P}$. Note that $\mathcal{P}$ consists precisely of completely monotone functions f satisfying $1/f \in \mathcal{BF}$.

Assume that $A \in \operatorname{Sect}(\omega)$ for some $\omega \in [0,\pi/2)$, and, in addition, that $\operatorname{ran}(A)$ is dense. Let f be a potential so that $f = 1/\psi$ for some $\psi \in \mathcal{BF}$. Then from the estimates of Bernstein functions in Proposition 2.3, (ii), (iii) and (iv) it follows that $\psi(A)$ and $f(A)$ can be defined by the extended holomorphic functional calculus using the regulariser e_ϵ given by

$$e_\epsilon(\lambda) := \left(\frac{\lambda}{(\epsilon+\lambda)(1+\epsilon\lambda)}\right)^2, \tag{4.11}$$

where $\epsilon > 0$ is fixed. Therefore, by [13, Proposition 1.2.2, d], we have

$$f(A) = [\psi(A)]^{-1}. \tag{4.12}$$

Moreover, for every h of the form $h = \psi + f$, where $\psi \in \mathcal{BF}$ and $f \in \mathcal{P}$, the (closed) operator $h(A)$ can also be defined by the extended holomorphic calculus via the regulariser e_ϵ, $\epsilon > 0$. Observe that from the product rule in Proposition 3.1, (i) it follows that $\mathrm{ran}(e_\epsilon(A)) \subset \mathrm{dom}(h(A))$, $\epsilon > 0$. Since $\mathrm{ran}(e_\epsilon(A)) = \mathrm{ran}(A^2) \cap \mathrm{dom}(A^2)$, $\epsilon > 0$, and the latter set is dense in X (see [20, Proposition 9.4, b) and c)]), the operator $h(A)$ is densely defined.

Lemma 4.9. *Assume that $h = \psi + f, \psi \in \mathcal{BF}, f \in \mathcal{P}$, and $A \in \mathrm{Sect}(\omega)$, $\omega \in [0, \pi/2)$. If A has dense range then*

$$\overline{\psi(A) + f(A)} = h(A).$$

Proof. If e_ϵ, $\epsilon > 0$, is given by (4.11), then

$$\lim_{\epsilon \to 0} e_\epsilon(A)x = x, \qquad x \in X,$$

and the statement follows from [13, Proposition 1.2.2, e)]. □

Now we can extend the class of admissible ψ in Theorem 4.5.

Corollary 4.10. *Suppose that $-A \in \mathcal{BH}(\theta)$ for some $\theta \in (0, \pi/2]$ and the range of A is dense. If $h = \psi + f$, where $\psi \in \mathcal{BF}$ and $f \in \mathcal{P}$, then $-h(A) \in \mathcal{BH}(\theta)$.*

Proof. First note that by Theorem 4.5 we have $-\psi(A) \in \mathcal{BH}(\theta)$. Moreover, by [13, Proposition 2.1.1, b)], inverses of generators of sectorially bounded holomorphic C_0-semigroups of angle θ generate semigroups of the same kind. Thus, by Theorem 4.5 and (4.12), $-f(A) \in \mathcal{BH}(\theta)$ as well.

From the product rule for the (extended) holomorphic functional calculus it follows that for every $s > 0$:

$$\begin{aligned}[(s + \psi(\cdot))^{-1}(s + f(\cdot))^{-1}](A) &= (s + \psi(A))^{-1}(s + f(A))^{-1} \\ &= (s + f(A))^{-1}(s + \psi(A))^{-1}.\end{aligned}$$

Hence, as in the proof of Lemma 4.6, the semigroups $(e^{-t\psi(A)})_{t\geq 0}$ and $(e^{-tf(A)})_{t\geq 0}$ commute. Then, by [8, Subsection II.2.7], $\overline{-\psi(A) - f(A)}$ generates the C_0-semigroup $(e^{-t\psi(A)}e^{-tf(A)})_{t\geq 0}$, and therefore $\overline{-\psi(A) - f(A)} \in \mathcal{BH}(\theta)$. From this, by Lemma 4.9, it follows that $-h(A) \in \mathcal{BH}(\theta)$. □

Note that in the particular case when $\psi \in \mathcal{CBF}$ and f is a Stieltjes function (i.e. $1/f \in \mathcal{CBF}$), Corollary 4.10 was proved in [3, Theorem 6.4].

5. Carasso–Kato functions

Let us first recall some notions and results from [6]. To this aim and for formulating our results in this section the next definition will be helpful.

A Bernstein function ψ is said to be *Carasso–Kato* if for every Banach space X, and every bounded C_0-semigroup $(e^{-tA})_{t\geq 0}$ on X, the C_0-semigroup $(e^{-t\psi(A)})_{t\geq 0}$ is holomorphic.

Following [6], denote the set of vaguely continuous convolution semigroups of subprobability Borel measures on $\mathbb{R}_+$ by $\mathcal{T}$. Let $\mathcal{I}$ stand for the set of $(\mu_t)_{t\geq 0} \in \mathcal{T}$ such that the Bernstein function ψ given by $(\mu_t)_{t\geq 0}$ via Bochner's formula (2.28) is Carasso–Kato. Let us finally denote by $\mathcal{T}_1 \subset \mathcal{T}$ the set of functions $\mathbb{R}_+ \ni t \mapsto \mu_t$ such that μ_t is continuously differentiable in $\mathrm{M}_b(\mathbb{R}_+)$ for $t > 0$, with $\|\mu_t{}'\|_{\mathrm{M}_b(\mathbb{R}_+)} = \mathrm{O}(t^{-1})$ as $t \to 0+$.

Recall that by [6, Theorem 4] one has $\mathcal{I} = \mathcal{T}_1$. Moreover, it was noted in [6, p. 872] that if $\psi \in \mathcal{BF}$ is given by $(\mu_t)_{t\geq 0} \in \mathcal{I}$, then

$$\psi(\mathbb{C}_+) \subset \Sigma_\gamma - \beta := \{\lambda \in \mathbb{C} : \lambda + \beta \in \Sigma_\gamma\} \tag{5.1}$$

for some $\gamma \in (0, \pi/2)$ and $\beta \geq 0$. Hence, as shown in [6, p. 873], there exists $K > 0$ such that

$$|\psi(\lambda)| \leq K|\lambda|^{2\gamma/\pi}, \qquad |\lambda| \geq 1, \qquad \lambda \in \mathbb{C}_+.$$

While [6] describes Carasso–Kato functions ψ in terms of the families of measures $(\mu_t)_{t\geq 0}$ corresponding to ψ via (2.28), the results of [6] are not so easy to apply since one is usually given ψ rather than the corresponding family $(\mu_t)_{t\geq 0}$. The aim of this section is to single out substantial classes of Carasso–Kato functions ψ in terms of geometric properties of ψ themselves.

Note first that if $\psi \in \mathcal{BF}$ and φ is Carasso–Kato then clearly $\varphi \circ \psi$ is Carasso–Kato. Corollary 4.7 and the composition rule (3.10) yield immediately that $\psi \circ \varphi$ is Carasso–Kato as well, and we separate this fact as the following corollary.

Corollary 5.1. *Let $\psi \in \mathcal{BF}$ and let φ be a Carasso–Kato function. Then $\psi \circ \varphi$ is also Carasso–Kato.*

Remark 5.2. Let $\psi, \varphi \in \mathcal{BF}$, so that

$$e^{-t\psi(\lambda)} = \int_0^\infty e^{-\lambda s}\,\mu_t(ds), \quad e^{-t\varphi(\lambda)} = \int_0^\infty e^{-\lambda s}\,\nu_t(ds), \quad \lambda \geq 0, \quad t \geq 0,$$

for some $(\mu_t)_{t\geq 0}$ and $(\nu_t)_{t\geq 0}$ from $\mathcal{T}$. Then, by [32, Theorem 5.27 and Lemma 13.3],

$$e^{-t(\psi\circ\varphi)(\lambda)} = \int_0^\infty e^{-\lambda\tau}\,\eta_t(d\tau),$$

where $(\eta_t)_{t\ge 0} \in \mathcal{T}$ is given by a convolution formula

$$\eta_t(d\tau) = \int_0^\infty \nu_s(d\tau)\, \mu_t(ds), \tag{5.2}$$

where the integral converges in the vague topology. Thus, if $\psi \in \mathcal{BF}$ and φ is Carasso–Kato then, in view of $\mathcal{I} = \mathcal{T}_1$, we infer that $(\eta_t)_{t\ge 0} \in \mathcal{T}_1$.

Example 5.3. *a*) It was proved in [6, Example 1] that the function $\varphi(\lambda) = \log(1+\lambda)$ is Carasso–Kato. Note that

$$e^{-t\log(1+\lambda)} = (1+\lambda)^{-t} = \int_0^\infty e^{-s\lambda} e^{-s} \frac{s^{t-1}}{\Gamma(t)}\, ds, \qquad t > 0.$$

Hence, by means of (2.28), the function φ corresponds to the semigroup of measures $(\nu_t)_{t\ge 0}$, where

$$\nu_t(ds) = \frac{s^{t-1}e^{-s}}{\Gamma(t)}\, ds, \qquad t > 0.$$

From Corollary 5.1 it follows that for every $(\mu_t)_{t\ge 0} \in \mathcal{T}$, the semigroup $(\eta_t)_{t\ge 0}$ given by

$$\eta_t(d\tau) = \left(\int_0^\infty \frac{\tau^{s-1}}{\Gamma(s)}\, \mu_t(ds)\right) e^{-\tau}\, d\tau, \qquad t > 0,$$

belongs to $\mathcal{T}_1$.

b) Consider the complete Bernstein function $\varphi(\lambda) = \sqrt{\lambda}$. Observe that

$$e^{-t\lambda^{1/2}} = \frac{t}{2\sqrt{\pi}} \int_0^\infty e^{-\lambda s} \frac{e^{-t^2/4s}}{s^{3/2}}\, ds, \qquad t > 0,$$

and it easy to check that φ is Carasso–Kato, see e.g. [6, Example 2]. By Corollary 5.1, for each $(\mu_t)_{t\ge 0} \in \mathcal{T}$ one has $(\eta_t)_{t\ge 0} \in \mathcal{T}_1$, where

$$\eta_t(d\tau) = \frac{1}{2\sqrt{\pi}} \left(\int_0^\infty s e^{-s^2/4\tau} \mu_t(ds)\right) \frac{d\tau}{\tau^{3/2}}, \qquad t > 0.$$

We proceed with several new conditions for a function to be Carasso–Kato. Roughly, they say that the function is Carasso–Kato if it shrinks an angular sector to a smaller one.

The first statement provides a geometric condition for a stronger version of the Carasso–Kato property. Recall that $\Sigma_\beta^+ = \{\lambda \in \mathbb{C} : 0 < \arg \lambda < \beta\}$.

Theorem 5.4. *Let $\psi \in \mathcal{BF}$. Suppose there exist $\theta_1 \in (0, \pi)$ and $\theta_2 \in (\pi/2, \pi)$, $\theta_1 < \theta_2$, such that ψ admits a holomorphic extension $\tilde{\psi}$ to Σ_{θ_2} and*

$$\tilde{\psi}(\Sigma^+_{\theta_2}) \subset \Sigma^+_{\theta_1}. \tag{5.3}$$

Then ψ is a Carasso–Kato function. Moreover, for any generator $-A$ of a bounded C_0-semigroup on X, one has

$$-\psi(A) \in \mathcal{BH}(\theta), \qquad \theta = \frac{\pi}{2}\left(1 - \frac{\theta_1}{\theta_2}\right).$$

Proof. Let

$$\alpha := \frac{\theta_2}{\pi} \in (1/2, 1) \qquad \text{and} \qquad \beta := \frac{\pi}{\theta_1} > 1.$$

For $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, define $g_\alpha(\lambda) := \lambda^\alpha$ and $g_\beta(\lambda) := \lambda^\beta$. Then, by (5.3), both functions $\tilde{\psi} \circ g_\alpha$ and $g_\beta \circ \tilde{\psi} \circ g_\alpha$ map the upper half-plane H^+ into itself. Hence, using Theorem 2.4, (iii), we conclude that

$$\tilde{\psi} \circ g_\alpha \in \mathcal{CBF} \qquad \text{and} \qquad g_\beta \circ \tilde{\psi} \circ g_\alpha \in \mathcal{CBF}. \tag{5.4}$$

Since $\pi/(2\alpha) < \pi$, Proposition 3.1, (ii) yields $A^{1/\alpha} \in \mathrm{Sect}(\pi/(2\alpha))$. Thus, in view of (5.4), the composition rule in Theorem 3.2, (ii) and Theorem 4.1 imply that

$$[\tilde{\psi} \circ g_\alpha](A^{1/\alpha}) = \left([g_\beta \circ \tilde{\psi} \circ g_\alpha](A^{1/\alpha})\right)^{1/\beta} \in \mathrm{Sect}(\pi/(2\alpha\beta)), \tag{5.5}$$

where the operators are defined by the Hirsch functional calculus.

Moreover, by Proposition 3.7, (ii),

$$\psi(A) = (\tilde{\psi} \circ g_\alpha)(A^{1/\alpha}).$$

Hence, using (5.5), we obtain that $\psi(A) \in \mathrm{Sect}(\frac{\pi}{2\alpha\beta})$. Since $\alpha\beta = \frac{\theta_2}{\theta_1}$, the required statement follows. □

Theorem 5.4 yields the following assertion providing a geometric condition for the Carasso–Kato property. The assertion is, in fact, Theorem 1.3 mentioned in Introduction.

Corollary 5.5. *Let $\psi \in \mathcal{BF}$. Suppose there exist $\theta \in (\pi/2, \pi)$ and $r > 0$ such that ψ admits a holomorphic extension $\tilde{\psi}$ to Σ_θ, and*

$$\tilde{\psi}(\lambda) \in \Sigma^+_{\pi/2} \quad \text{for} \quad \lambda \in \Sigma^+_\theta, \quad |\lambda| \ge r.$$

Then ψ is a Carasso–Kato function. Moreover, for any generator $-A$ of a bounded C_0-semigroup on X, one has $-\psi(A) \in \mathcal{H}(\frac{\pi}{2}(1 - \frac{\pi}{2\theta}))$.

Proof. From our assumptions it follows that there exists $d > 0$ such that the Bernstein function ψ_d given by

$$\psi_d(\lambda) := \psi(\lambda + d), \qquad \lambda > 0,$$

admits a holomorphic extension $\tilde{\psi}_d$ to Σ_θ, and

$$\tilde{\psi}_d(\lambda) \in \Sigma^+_{\pi/2} \quad \text{for} \quad \lambda \in \Sigma^+_\theta.$$

Therefore, by Theorem 5.4 with $\theta_2 = \theta$ and $\theta_1 = \pi/2$, if $-A$ is the generator of a bounded C_0-semigroup then $-\psi_d(A) = -\psi(A + d) \in \mathcal{BH}(\theta_0)$, where

$$\theta_0 = \frac{\pi}{2}\left(1 - \frac{\pi}{2\theta}\right) \in \left(0, \frac{\pi}{2}\right).$$

Then, by Lemma 4.6, we conclude that $-\psi(A)$ generates a bounded C_0-semigroup possessing a holomorphic extension to Σ_{θ_0}. $\square$

Note that Corollary 5.5 generalizes and improves Fujita's conditions from [9, p. 337 and Lemma 2] ensuring that a Bernstein function ψ is Carasso–Kato. In particular, Corollary 5.5 shows that one can omit Fujita's condition (A3) and his assumption that ψ is regularly varying. Indeed, Fujita's assumptions (A1) and (A2) imply that there exist $\alpha \in (0, 1)$ and $\theta_\alpha \in (\pi/2, \pi/(2\alpha))$ such that for a sufficiently large $r > 0$ the function ψ has a holomorphic extension $\tilde{\psi}$ to $\Sigma^+_{\theta_\alpha}$, and

$$\tilde{\psi}(\lambda) \in \Sigma^+_{\alpha\theta_\alpha} \subset \Sigma^+_{\pi/2} \quad \text{if} \quad \lambda \in \Sigma^+_{\theta_\alpha}, \quad |\lambda| \geq r,$$

so that the assumptions of Corollary 5.5 are satisfied.

Let us illustrate Corollary 5.5 by the next example.

Example 5.6. To simplify our notation in this example we identify functions with their holomorphic extensions.

Consider the Bernstein function $f(\lambda) = \lambda + 1 - e^{-\lambda}$, $\lambda > 0$. If $\lambda = re^{i\beta}$, $\beta \in [0, \pi/2]$, then

$$\mathrm{Im}(f(\lambda)) = r \sin\beta + e^{-r\cos\beta} \sin(r \sin\beta) \geq (1 - e^{-r\cos\beta}) r \sin\beta \geq 0.$$

Let $\alpha \in (1/2, 1)$ be fixed. Then the function

$$f_\alpha(\lambda) := f(\lambda^\alpha) = \lambda^\alpha + 1 - e^{-\lambda^\alpha}, \qquad \lambda > 0, \tag{5.6}$$

is Bernstein as the composition of Bernstein functions. Moreover, f_α extends holomorphically to $\mathbb{C} \setminus (-\infty, 0]$. Since $f_\alpha(\lambda) \in \Sigma^+_{\pi/2}$ for $\lambda \in \Sigma^+_{\pi/(2\alpha)}$, the function f_α satisfies the assumptions of Corollary 5.5 with $\theta = \frac{\pi}{2\alpha} \in (\pi/2, \pi)$. Hence, f_α is Carasso–Kato. (Note that on the other hand, in view of (5.6), one may apply the composition rule (3.10) and Theorem 4.5 to arrive at the same conclusion.)

Let now

$$\psi(\lambda) := \log(1+\lambda), \qquad g_\alpha(\lambda) := f_\alpha(\lambda)\cdot\psi(\lambda^{1-\alpha}), \quad \lambda \in \mathbb{C}\setminus(-\infty,0],$$

where one chooses the main branch of the logarithm. Clearly, g_α is holomorphic in $\mathbb{C}\setminus(-\infty,0]$. Moreover, g_α is Bernstein by [32, Proposition 3.8, (vi)].

Let $\gamma \in (\alpha,1)$ be fixed. Observe first that $f_\alpha(\lambda) \in \Sigma^+_{\pi/2}$ for $\lambda \in \Sigma^+_{\pi\gamma/(2\alpha)}$. On the other hand, from Proposition 2.2 it follows that $f_\alpha(\lambda) \in \Sigma_{(\pi/2)\gamma}$ if $\lambda \in \Sigma_{\pi\gamma/(2\alpha)}$. So, $f_\alpha(\lambda) \in \Sigma^+_{(\pi/2)\gamma}$ for $\lambda \in \Sigma^+_{\pi\gamma/(2\alpha)}$. Finally, note that $\mathrm{Im}(\psi(\lambda)) \in (0,\pi)$ for $\lambda \in H^+$ and $\psi(\lambda) \in \mathbb{C}_+$ for $\lambda \in \mathbb{C}\setminus(-\infty,0]$ such that $|\lambda| \ge 2$. Summarizing the observations above, we infer that

$$g_\alpha(\lambda) \in \mathbb{C}_+ \quad \text{if} \quad \lambda \in \Sigma^+_{\pi\gamma/(2\alpha)}, \quad |\lambda| \ge r_{\alpha,\gamma},$$

where

$$r_{\alpha,\gamma} := \max\left(2^{1/(1-\alpha)}, \left(1+\frac{\pi}{\cos(\pi\gamma/2)}\right)^{1/(1-\alpha)}\right) = \left(1+\frac{\pi}{\cos(\pi\gamma/2)}\right)^{1/(1-\alpha)}.$$

So, g_α satisfies the assumptions of Corollary 5.5 with $\theta = \frac{\pi\gamma}{2\alpha}$, and then g_α is Carasso–Kato.

Note that $g_\alpha \notin \mathcal{CBF}$ since it does not have a sublinear growth in $\Sigma_{\frac{\pi\gamma}{2\alpha}}$. Moreover, since

$$g_\alpha(\lambda) = 2\lambda + \mathrm{O}(|\lambda|^{2-\alpha}), \quad \lambda \to 0,\ \lambda \in \mathbb{C}_+,$$

g_α is not of the form $u(\lambda^\beta)$, for $u \in \mathcal{BF}$ and $\beta \in (0,1)$, and Theorem 4.5 is not applicable to g_α.

Now we turn our attention to Carasso–Kato functions ψ which are, in addition, *complete* Bernstein functions. As in the situation of Theorem 5.4, we first require ψ to map the generators of bounded C_0-semigroups into the generators of *sectorially bounded* holomorphic C_0-semigroups. Such ψ can, in fact, be *characterized* in an elegant way as the following statement shows. (It corresponds to Theorem 1.2 from Introduction.)

Theorem 5.7. *Let $\psi \in \mathcal{CBF}$ and let $\gamma \in (0,\pi/2)$ be fixed. The following assertions are equivalent.*

(i) *One has*

$$\psi(\mathbb{C}_+) \subset \Sigma_\gamma.$$

(ii) *For each Banach space X and each generator $-A$ of a bounded C_0-semigroup on X, the operator $-\psi(A)$ belongs to $\mathcal{BH}(\pi/2-\gamma)$.*

Proof. The implication (ii) $\Rightarrow$ (i) follows from [6, Theorem 4] and its proof. So, it suffices to prove that (i) implies (ii).

Assume that (i) is true. We can also assume that $\psi \not\equiv$ const. Then, by Theorem 5.4 and Proposition 2.5, we obtain that if $\theta_0 \in (\pi/2, \pi)$ is given by

$$|\cos\theta_0| = \frac{\cot\gamma}{\cot\gamma + 1}$$

then for every $\theta \in (\pi/2, \theta_0)$ one has

$$-\psi(A) \in \mathcal{BH}(\omega), \quad \omega = \frac{\pi}{2}\left(1 - \frac{\tilde{\theta}}{\theta}\right), \tag{5.7}$$

where $\tilde{\theta} = \tilde{\theta}(\theta) \in (0, \pi/2)$ is defined by

$$\cot\tilde{\theta} = \frac{\cot\gamma + 1}{\sin\theta}\left(\frac{\cot\gamma}{\cot\gamma + 1} - |\cos\theta|\right).$$

Since

$$\lim_{\theta\to\pi/2} \tilde{\theta}(\theta) = \gamma,$$

considering θ in (5.7) arbitrarily close to $\pi/2$, we obtain (ii). □

Let us recall that (5.1) is necessary for $\psi \in \mathcal{BF}$ to be a Carasso–Kato function. The next statement shows that if moreover $\psi \in \mathcal{CBF}$ then (5.1) is also sufficient thus providing a characterization of the Carasso–Kato property for complete Bernstein functions.

Corollary 5.8. *Let $\psi \in \mathcal{CBF}$. Then ψ is Carasso–Kato if and only if there exist $\gamma \in (0, \pi/2)$ and $\beta \geq 0$ such that (5.1) holds. Moreover, if (5.1) holds and if $-A$ generates a bounded C_0-semigroup on X, then $-\psi(A) \in \mathcal{H}(\pi/2-\gamma)$.*

Proof. It is sufficient to show that (5.1) implies that ψ is Carasso–Kato. If (5.1) is satisfied, then set $\psi_\beta(\lambda) := \psi(\lambda + \beta)$ and note that $\psi_\beta \in \mathcal{CBF}$ for each $\beta \geq 0$. By applying Theorem 5.7 to ψ_β, we obtain that

$$-\psi_\beta(A) = -\psi(\beta + A) \in \mathcal{BH}(\pi/2-\gamma)$$

for any generator $-A$ of a bounded C_0-semigroup on X. Then, using Lemma 4.6, we conclude that $-\psi(A)$ generates a bounded C_0-semigroup having a holomorphic extension to $\Sigma_{\frac{\pi}{2}-\gamma}$. □

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Appendix A

It is an open question whether for any $\alpha \in (0,1)$,

$$\psi \in \mathcal{BF} \Longrightarrow [\psi(\lambda^\alpha)]^{1/\alpha} \in \mathcal{BF}. \tag{A.1}$$

A positive answer to this question would allow one to apply the methods from [3] directly and to obtain Theorem 4.5 and its corollaries in a comparatively simple way. Let us analyze the property (A.1) in some more details.

Apart from the situation described in (4.2), it is known that (A.1) is true if $\alpha = 1/n$, $n \in \mathbb{N}$ [3, Proposition 7.1] (see also [28, Remark 12]). The following Proposition A.2 generalizes [3, Proposition 7.1] and (4.2) in the case when $\alpha \in (0,1/2]$ and $\psi \in \mathcal{BF}$, and it extends these statements for any $\alpha \in (0,1)$ if ψ is a so-called special Bernstein function. Recall that a non-zero $\psi \in \mathcal{BF}$ is said to be a special Bernstein function, if $\lambda/\psi(\lambda) \in \mathcal{BF}$ as well. The class of special Bernstein functions will be denoted by $\mathcal{SBF}$. By Theorem 2.4, (iii) we have $\mathcal{CBF} \subset \mathcal{SBF}$.

The proof of the proposition is based on the next lemma.

Lemma A.1. *Let $\psi \in \mathcal{BF}$, and let $\alpha \in (0,1)$. For $\beta > 0$ define*

$$\psi_{\alpha,\beta}(\lambda) := \left(\frac{\psi(\lambda^\alpha)}{\lambda^\alpha}\right)^\beta, \qquad \lambda > 0. \tag{A.2}$$

Then $\psi_{\alpha,\beta}$ is completely monotone for all $\alpha \in (0,1/2]$ and $\beta > 0$. If $\psi \in \mathcal{SBF}$, then $\psi_{\alpha,\beta}$ is completely monotone for all $\alpha \in (0,1)$ and $\beta > 0$.

Proof. If $\psi \in \mathcal{BF}$ and $\alpha \in (0,1/2]$, then by [32, Proposition 7.22] one has

$$f_\alpha(\lambda) := \lambda^{1-\alpha}\psi(\lambda^\alpha) \in \mathcal{CBF},$$

so that $\lambda/f_\alpha(\lambda) \in \mathcal{CBF}$ and, by [32, Theorem 3.7, (ii)], for any $\beta > 0$,

$$\psi_{\alpha,\beta}(\lambda) = \left[\frac{\lambda^{1-\alpha}\psi(\lambda^\alpha)}{\lambda}\right]^\beta = \left[\frac{\lambda}{f_\alpha(\lambda)}\right]^{-\beta}$$

is completely monotone. Let now $\alpha \in (0,1)$ and $\psi \in \mathcal{SBF}$ so that $\psi(\lambda) = \lambda/f(\lambda)$, $f \in \mathcal{BF}$. Since the set $\mathcal{BF}$ is closed under composition, $f(\lambda^\alpha)$ is a Bernstein function. Hence, by [32, Theorem 3.7, (ii)], the function $\psi_{\alpha,\beta}(\lambda) = [f(\lambda^\alpha)]^{-\beta}$ is completely monotone for any $\beta > 0$ as the composition of completely monotone and Bernstein functions. □

Proposition A.2. *Let $\psi \in \mathcal{BF}$, and let $\psi_\alpha(\lambda) := [\psi(\lambda^\alpha)]^{1/\alpha}$, $\alpha \in (0,1)$. If $\alpha \in (0,1/2]$, then $\psi_\alpha(\lambda) \in \mathcal{BF}$. If $\alpha \in (0,1)$ and, in addition, $\psi \in \mathcal{SBF}$, then $\psi_\alpha \in \mathcal{SBF}$ too.*

Proof. For $\alpha \in (0,1)$ and $\beta_\alpha = 1/\alpha - 1$ consider $\psi_{\alpha,\beta_\alpha}$ given by (A.2). By Lemma A.1, if $\psi \in \mathcal{BF}$ and $\alpha \in (0,1/2]$ then $\psi_{\alpha,\beta_\alpha}$ is completely monotone.

Observe further that

$$\psi'_\alpha(\lambda) = \psi'(\lambda^\alpha)\psi_{\alpha,\beta_\alpha}(\lambda), \qquad \lambda > 0. \tag{A.3}$$

If $\psi \in \mathcal{BF}$, then $\psi'(\lambda^\alpha)$ is completely monotone being the composition of completely monotone and Bernstein functions. Hence, since the product of completely monotone functions is completely monotone, we infer that ψ'_α is completely monotone for every $\alpha \in (0,1/2]$. Thus $\psi_\alpha \in \mathcal{BF}$.

If, in addition, $\alpha \in (0,1)$ and $\psi \in \mathcal{SBF}$ then, by Lemma A.1, the function $\psi_{\alpha,\beta_\alpha}$ is completely monotone. Hence, arguing as above, from (A.3) it follows that in this case ψ'_α is completely monotone, and then $\psi_\alpha \in \mathcal{BF}$. □

Example A.3. Note that (A.1) does not imply that $\psi \in \mathcal{SBF}$. For instance, if

$$\psi(\lambda) := 1 - \frac{1}{(1+\lambda)^2} = \frac{\lambda(2+\lambda)}{(1+\lambda)^2}, \qquad \lambda > 0,$$

then $\psi \in \mathcal{BF}$ since $\psi'(\lambda) = 2(1+\lambda)^{-3}$ is completely monotone. At the same time, $\psi \notin \mathcal{SBF}$ since $(\lambda/\psi(\lambda))' = 1 - 1/(2+\lambda)^2$ is not completely monotone.

On the other hand, we have

$$\psi(\lambda) = \frac{\lambda}{f_1(\lambda)f_2(\lambda)},$$

where

$$f_1(\lambda) = 1 + \lambda \in \mathcal{CBF}, \qquad \text{and} \qquad f_2(\lambda) = \frac{1}{2} + \frac{\lambda}{2(2+\lambda)} \in \mathcal{CBF}.$$

Thus, if $\alpha \in (0,1)$, then

$$([\psi(\lambda^\alpha)]^{1/\alpha})' = \frac{2}{(1+\lambda^\alpha)^3} \cdot [f_1(\lambda^\alpha)]^{-(1/\alpha-1)} \cdot [f_2(\lambda^\alpha)]^{-(1/\alpha-1)}$$

is completely monotone. Indeed, $f_j(\lambda^\alpha) \in \mathcal{BF}$, $j = 1,2$, and from [32, Theorem 3.7, (ii)] it follows that $[f_j(\lambda^\alpha)]^{-\beta}$, $j = 1,2$, are completely monotone for $\beta > 0$. Hence $([\psi(\lambda^\alpha)]^{1/\alpha})'$ is completely monotone as the product of completely monotone functions, and then $[\psi(\lambda^\alpha)]^{1/\alpha} \in \mathcal{BF}$.

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